1-1-2003

# A sixth grade student's geometric thinking: the van Hiele levels of geometric thought-constructivist approach to the phases of instruction

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# A sixth grade student's geometric thinking: The van Hiele levels of geometric thought-Constructivist approach to the phases of instruction

by

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A thesis submitted to the graduate faculty

in partial fulfillments of the requirements for the degree of

MASTER OF SCIENCE

Major: Education

Program of Study Committee:
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Iowa State University

Ames, Iowa

2003

Graduate College

Iowa State University

This is to certify that the master's thesis of

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Signatures have been redacted for privacy

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# ABSTRACT

Geometric shapes around us are the initial conceptions for mathematical thinking. Geometric thought develops throughout the interval from a concrete level of geometric understanding to an abstract level of geometric perception. Many studies have described students' geometric understanding using the van Hiele levels of geometric thinking, which help the teacher to understand his/her students' development of geometric concepts. The purpose of this case study was to explore a sixth grade student's geometric thinking based upon the van Hiele levels of geometric thought, in relation to his attitudes toward mathematics and to discuss whether or not it is a constructivist approach to use the phases of instruction of van Hiele levels of geometric thinking.

# INTRODUCTION

Many mathematics educators (Ball, 1993; Cobb, Yackel, & Wood, 1992a, 1992b; Confrey, 1990; Kaput, 1987, 1991; Simon, 1995; Thompson, 1989) have recently focused on how individuals develop particular mathematical knowledge (Thompson, 1994), what kind of cognitive activities students show to represent their thinking (Cobb et al., 1992a), and how teachers can anticipate children's mathematical thinking and learning processes (Carpenter & Moser, 1983; Simon, 1995). Mathematics educators who have developed theories in mathematics education view learning as "a process of constructing internal mental representations" (Cobb et al., 1992a, p. 2), "a process of individual and social construction" (Simon, 1995, p. 117), or as "a process of conceptual change" (Posner, Strike, Hewson, and Gertzog, 1982, p. 212).

The common point that all these views meet is that learning and knowing are approached from the perspective of the individual learner. Constructivism postulates that the student is not a passive consumer of knowledge exposed by the teacher. Rather he/she is an active learner who constructs his/her own understanding of content by interacting with what he/she currently knows in the learning environment (Hewson, 1992; Posner et al., 1982; Schunk, 2000; Scott, 2001; Simon, 1995). Cobb et al. (1992a) argue that as students develop *external representations*[1] and use ordinary symbols to express their thinking, they modify these processes by their cognitive representations. Taking this argument into account, knowledge is constructed by learners with the interaction between external relationships in a sociocultural context and internal structure shaped by their prior knowledge and mathematical reasoning skills (Cobb et al., 1992a; Henriques, 1997; Schunk, 2000). Simon

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[1] von Glasersfeld (1987) separates the term representation; internal representations are located in students' heads and external representations are located in the environment.

(1995) mentions the fact that, from a philosophical position, there is a reality independent of our way of knowing it. Furthermore, he emphasizes that people form their knowledge of the outside world from their perceptions and experiences. In this concept of reality, there is another reality – that is, students' reality – that has been shaped in the sociocultural context. "*Students' mathematics*[2] is something we attribute to students independently of our interactions with them" (Steffe & Thompson, 2000, p. 268). Understanding how students think mathematically helps the teacher specify learning goals and form learning processes using *hypothetical entries*[3]. Cobb et al. (1992a) discuss the role of the teacher to emphasize that "social interaction plays an important role in students' mathematical learning" (p. 5). Consequently, students learn not only through their experiences but they also learn from the teacher in a dialogue. Moreover, Cobb et al. (1992a) characterized the teacher in the position of instructor focusing on three major features:

i. Students are facilitated to build mathematical relationships that are placed outside their minds – a general goal of the instruction.

ii. Students should be provided with instructional representations that help them to constitute internal relationships to reach this instructional goal.

iii. Students use the external instructional materials to construct their mental mathematical knowledge.

Cobb et al. (1992a) point out that characterizing mathematical learning has a two-dimensional concern when constructivism is merged with the representational view: the students' and the teacher's interpretations of instructional representations. On the one hand,

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[2] "Students' mathematics" refers to whatever might constitute students' mathematical realities; "mathematics of students" refers to the interpretation of students' mathematics.

[3] Hypothetical entry refers to teacher's hypotheses of students' mathematical understanding and those of their mathematics knowledge.

learning is viewed as a process of students' construction of mathematical knowledge during the attempt to make sense of their environment (Borasi, 1992; Posner et al., 1982; Reusser, 2000; Simon, 1995). On the other hand, learning can be described as a process in which students recognize mathematical relationships presented in instructional representations (Cobb et al., 1992a). These two dimensions show a slight difference in terms of learners' and teacher's interpretations. The first one focuses on students' actively constituting their own mathematical ways of knowing and their interpretations of instructional representations. Posner et al. (1982) view learning as a process of conceptual change in which students construct their mathematical knowledge on the basis of their current ideas and the interaction with what they are taught. On the other side of the coin, instruction is more based on teacher's interpretations, and students' recognition of mathematical relationships posed by the teacher.

Learning is a two-way interaction between the learner's cognitive structure and external representations, mostly introduced by the teacher. Instead of directly introducing mathematical concepts to pupils, students should have the opportunity to reconstruct the content through inquiry (De Villiers, 1998). From a constructivist standpoint, students should be involved actively in the learning process of defining geometric concepts rather than learning definitions (Cobb et al., 1992; De Villiers, 1998). The National Council of Teachers of Mathematics (NCTM) (2000) aims that through instruction of K-12 school geometry students will:

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;

- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations; and
- use visualization, spatial reasoning, and geometric modeling to solve problems (p. 41).

Geometric understanding develops through the transition from the concrete level of understanding to the abstract level of understanding the geometric shapes (Arf, 1960). This transition is completed when geometric shapes are conceptualized and abstracted by the individual throughout his/her experiences with the geometric shapes. Children's understanding of geometric shapes involves touching one side of an object and understanding the part, then touching other sides to grasp what they look like, then combining these parts, and finally conceptualizing the shape (Clements & Battista, 1992). Understanding the geometry around us is interrelated with geometric concepts.

Children's understanding of geometric concepts can be described using the van Hiele levels of geometric thought. The van Hieles formulated a theory of geometric thinking, which has five sequential levels – visualization, analysis, abstraction, deduction, and rigor – whose achievement depends upon the accomplishment of the previous levels. According to this theory, students progress through levels of thinking in a hierarchy. To van Hiele, progress from one level to the next is independent of age; rather, it is more related to the instruction.

The purpose of this case study is to explore a sixth grade, male student's understanding of geometric concepts, based on van Hiele levels of geometric thought; whether his attitude toward mathematics changes through the levels; and how the learning sessions we had reflect constructivist views of learning to van Hiele levels and geometry. Thus, this study has three aspects: (1) his attitude toward mathematics even at the different van Hiele levels from my observations; (2) an analysis of a sixth grade male student's demonstrations of his geometric thinking based on van Hiele levels of geometric thought; and (3) a discussion of how the data (learning episodes) mirror the constructivist approach to van Hiele levels and geometry.

# LITERATURE REVIEW

This section consists of four main parts: (1) conceptual change model and mathematics teaching cycle, where I describe constructivism in mathematics education; (2) learning geometry, where I focused on geometric understanding and thinking according to the van Hiele levels of thought; (3) implementations of constructivism in geometry; and (4) relationships between affective domain and mathematics achievement.

## Conceptual Change Model and Mathematics Teaching Cycle

Constructivism, partly based on the work of Swiss developmental psychologist Jean Piaget (1896-1980), is a cognitive learning theory postulating that learners are actively involved in the learning process and construct their own knowledge for themselves by making sense of understanding, where *knowledge, learning, and understanding* refers to more than just recalled *facts* (Hewson, 1992; Posner, Strike, Hewson, & Gertzog, 1982; Schunk, 2000; Scott, 2001; Simon, 1995). From this point of view, learning and knowing are approached from the perspective of the individual. "The main focus is on understanding what learning is, not what learning depends on" (Posner et al., 1982, p. 211). Henriques (1997) and Schunk (2000) state that Vygotsky emphasizes further on social environment in learning. Taking this view into account, knowledge is constructed by learners driving from the interactions with the natural world in a sociocultural context and their prior knowledge (Henriques, 1997; Schunk, 2000). Suggesting children's active participation in the development of their understanding, constructivism as a learning theory has been widely used in mathematics learning.

From a philosophical position, there is a "fact" ("real world" and/or "objective reality") independent of observer (knower) (Scott, 2001; Simon, 1995). This reality is "independent of our way of knowing it. ... We construct our knowledge of our world from our perceptions and experiences," using our practical world in the learning process to make sense of our perceptions (Simon, 1995, p. 115). According to Scott (2001), the key domain of "science" is the approval of the "objective hypothesis," that is, as stated above, "there is no way of knowing whether a concept fits with an objective reality." In this concept of "reality," there is another reality – which is students' reality – that has been shaped in the sociocultural context. "*Students' mathematics* is something we attribute to students independently of our interactions with them" (Steffe & Thompson, 2000, p. 268). Learning emerges "as an independent contribution" of the interacting students (p. 287).

Posner et al. (1982) view learning as "a process of conceptual change" (p. 212). This process contains the product of what the student is taught and his/her current ideas or concepts. It has been accepted that students have initial ideas (i.e., they can compare sets at least qualitatively by stating which one is bigger) before the formal learning setting (Reusser, 2000). Conceptual change has "links to constructivism (view of how people learn) and to students' conceptions (alternative conceptions or 'mis'conceptions)" (Hewson, 1992; Hewson & Hewson, 1992). Using the idea of Piaget's accommodation and assimilation, Posner et al. (1982) poses an extended model, *the conceptual change model,* to explore how people learn. More specifically, children bring their beliefs and ideas, some of which are not scientifically/mathematically accepted, into the learning environment. Ausubel (1968) suggested that the learner's existing ideas, concepts, emotions, and beliefs – "cognitive structure" – into which new information or concept can be embodied, help the new information to be more likely accepted.

According to Piaget, for cognitive development, individuals receive conflicting environmental input (Schunk, 2000). To overcome this conflict, there are two possible processes that one can track: assimilation and accommodation. To Posner et al., these two processes are the phases of conceptual change. The first phase starts with the problems defined by the central *commitments*[4], whereby the student continues finding strategies for coping with them, and eventually ends up with certain standards for acceptable solutions. The second phase of conceptual change depends on the modification of these central commitments. This modification compels the person to feel his or her current ideas and conceptions are inadequate to deal with the new phenomena. If students use their existing ideas or concepts to cope with this new situation, this is a variant of what Posner et al. call *assimilation*. If students' current concepts cannot cope with the new phenomenon, then students need radical changes or modifications in their central commitments: *accommodation*[5]. In addition, Schunk (2000) defines assimilation as the process of fitting the incoming information to the existing cognitive structure, and accommodation as the process of changing internal structures to make sense of the reality. "As reality is assimilated, structures are accommodated" (p. 234).

Driver and Oldham (1986) draw on Rumelhart and Norman (1981) to explain how cognitive structures may change: *accretion* – addition to an existing one, *tuning* – modification, and *restructuring* – radical changes. Posner et al. (1982) suggest four conditions (*dissatisfaction, intelligibility, plausibility*, and *fruitfulness*), to promote successful

[4] In Posner et al.'s article the terms "commitments," "concepts," and "conceptions" are equivalent (p. 212).
[5] These are Piaget's words. Posner et al. use these words, assimilation and accommodation, to be explicit, jotting down this explanation.

accommodation, the radical form of conceptual change. Consequently, Driver and Oldham (1986) view learning as the way conceptions change using exploratory language.

Dissatisfaction, intelligibility, plausibility, and fruitfulness (Posner et al., 1982) are the four proposed conditions that must be completed prior to an ultimate accommodation. To understand better the conditions of accommodation, think of a situation or anomaly that has four phases in succession. The first phase occurs when one encounters an unsolvable problem and realizes that his/her existing concepts are not enough to solve this problem: *dissatisfaction.* Now, he/she needs new information/conception to resolve the puzzle. This step requires this new conception to be *intelligible* and *plausible;* these are the next two phases. This new intelligible and plausible conception will help the individual solve the problem. The last phase of accommodation occurs at this point where the new conception is "open up new areas of inquiry": *fruitfulness* (p. 214).

### *Implementations of Constructivism in Curriculum*

Many scholars (Bell, 1985; Brook & Driver, 1986; Osborne & Wittrock, 1983; Pope & Gilbert, 1983; Wightman, 1986) have been trying to apply constructivist learning theory to curriculum. Driver and Oldham (1986) modified a constructivist approach to curriculum development in the science content area.

Each learning theory has its own curriculum design. Driver and Oldham developed a curriculum model based on the constructivist view of learning, which changes the roles of teacher and students. Contrary to the traditional view of learning, where the learner is a passive consumer of information and the teacher is an active information donor; from the constructivist view of learning, the learner is the center of learning, and the only one who constructs the knowledge with the interaction of what he or she brings and what is provided from the learning environment. Radical constructivist Ernst von Glasersfeld (1999) highlighted the role of the teacher with the answer of the question asked by Carla DeLancy via electronic communication:

> ...They (*students*) must, indeed, construct their knowledge themselves, and if it cannot be measured by a comparison with some objective "truth", it will be good knowledge only if it works in the experiential world as well as the teacher's or better. In my view, teachers can only SHOW how knowledge could be constructed, they can never transfer what they happen to know. But this, of course, presents a challenge that many are afraid of (Ernst von Glasersfeld's Answers, April 1999, http://www.oikos.org/vonanswerapril.htm [Italics are added]).

Moreover, others (Henriques, 1997; Matthews, 1994; Schunk, 2000; Scott, 2001; Shymansky, 1994; Sivertsen, 1993; Yore, 2001), approaching from an interactive perspective, state that the teacher is not an expert who has authority of knowledge and who tells the students what they need to know; rather he/she is a facilitator who fosters conceptual knowledge by helping students to make plausible and construct themselves in a conversation.

In a constructivist curriculum model, there are four major inputs into curriculum design:

- *Content*, which contains the scientific ideas to which the students are to be exposed.
- *Students' prior knowledge*, which they bring into the learning environment, and which should not be ignored.

- *Perspective on the learning process*, which shapes the learning design (from a constructivist curriculum model; this is a conceptual change model and a constructivist learning view).
- *Teacher's practical knowledge of students*, schools, and classrooms, which helps to deal with possible problems and constraints.

Driver and Oldham suggest that the curriculum design – design of learning strategies and materials – can be shaped by these inputs. They also view curriculum as "the set of learning experiences which enable the learners to develop their understanding" (p. 112). Accomplishment of learning strategies and materials in classrooms and the evaluation of this accomplishment complete the constructivist curriculum model.

Driver and Oldham (1986) also give a teaching model based on the constructivist view of learning. "Constructivist teaching," if it is considered to be teaching even though it is a learning theory (Simon, 1995), is likely to be a term referring to teaching that is informed by constructivism, e.g., whose main focus is that a student uses his/her current knowledge to construct new knowledge (Selden & Selden, June 1998, http://www.maa.org/t_and_l/sampler/rs_glossary.html). This teaching model (sequence) starts with students' expression of their ideas and bringing them out, passes through the conceptual change process to the completion of assimilation or accommodation, and ends up with review of changes in ideas and comparison with previous ideas (Driver & Oldham, 1986). In this teaching sequence, the role of teacher is to provide environments that engage individual students and develop their explanation and communication skills and to prepare challenges that communicate student perceptions and interpretations. To Steffe and

Thompson (2000), another important role of teacher is "to bring students' spontaneous schemes to foster students' successful assimilation" (p. 289).

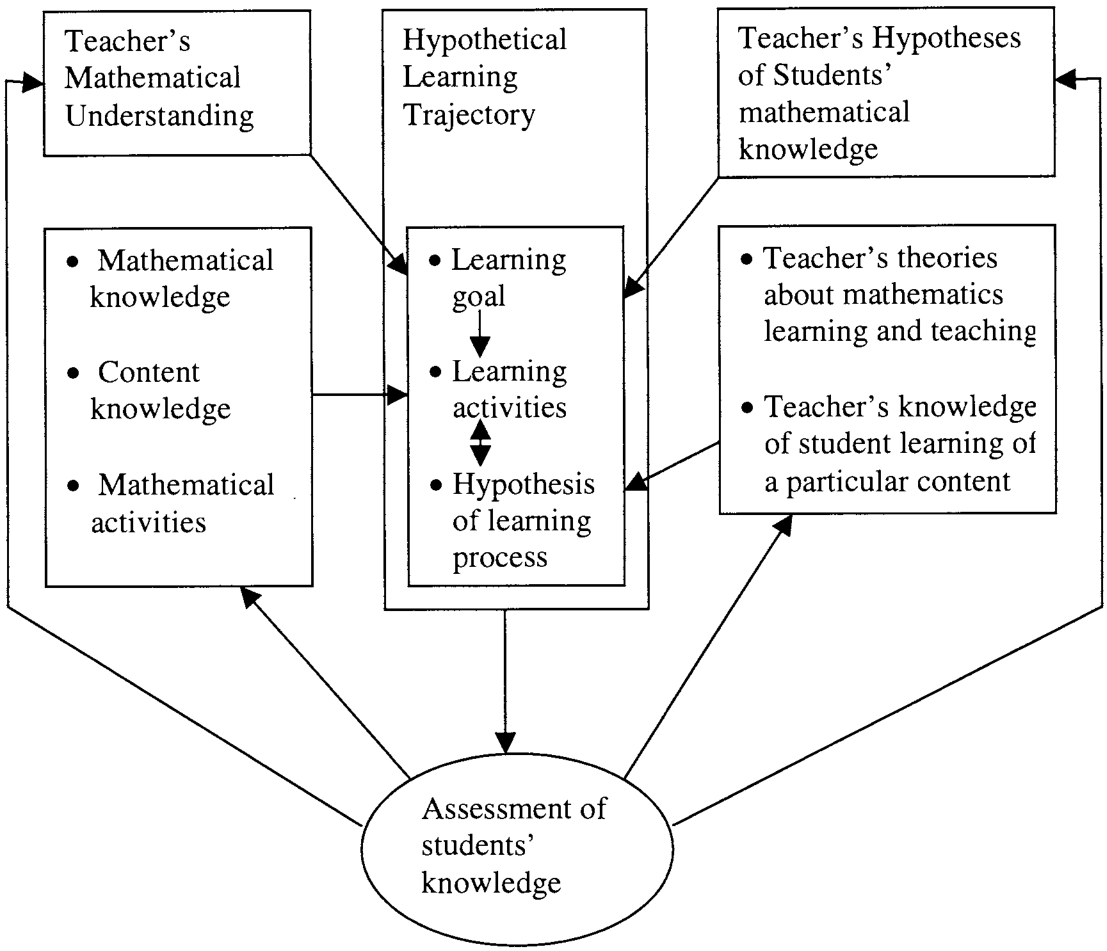



Figure 2.1. Mathematics teaching cycle (adapted from Simon, 1995).

Another constructivist teaching model was developed by Simon (1995), especially in the mathematics content area: mathematics "teaching cycle" (see Figure 2.1). This model can be integrated into any content area concerning a constructivist learning base. According to this model, there are two main factors that shape design of the lesson: "the teacher's mathematical understanding – content knowledge of teacher – and the teacher's hypotheses about the students' mathematical knowledge" (p. 135). As Steffe and Thompson (2000) and

Simon (1995) mention, the teacher's assumptions of students' mathematical understanding are structured by his/her understanding of mathematical content and the mathematics of the students. Figure 2.1 shows that these two factors are interconnected, and interactive through the teacher's learning goal.

This learning goal is hypothetical because the learning process is regulated according to the interaction between teacher's content knowledge and teacher's hypotheses of students' mathematics understanding. This learning process is "*hypothetical learning trajectory*" (Simon, 1995, p. 135). The hypothetical learning trajectory consists of three components – the learning goal that determines the path, the learning activities, and hypothetical learning process (of students) – which may need modification through the class period. Teacher compares his/her mathematical understanding with his/her hypotheses about students' knowledge, and decides the big ideas that are generated through the learning trajectory. Continuous assessment of students' conceptual understanding can cause modifications in the teacher's mathematical understanding and the teacher's hypotheses of students' mathematical understanding that, in turn, lead to a new learning trajectory (Simon, 1995).

Mathematics educators (Hiebert & Carpenter, 1992; Simon, 1995; Van de Walle, 1998) define knowledge *of* mathematics as conceptual and procedural knowledge of the subject and *about* mathematics as "understanding about the nature of mathematical knowledge and activity: what is entailed in doing mathematics and how truth is established in the domain" (Ball, 1991, p. 7). Conceptual knowledge is rich knowledge in relationships constructed internally in connected network (Hiebert & Carpenter, 1992; Van de Walle, 1998). Procedural knowledge is a series of actions, rules, and procedures that one follows to accomplish some mathematical tasks. The connections of procedural knowledge are the automatized internal representations between succeeding actions (Goetz, Alexander, & Ash, 1992; Hiebert & Carpenter, 1992; Van de Walle, 1998). The relationships between conceptual knowledge and procedural knowledge can be stated as:

> If the learner connects the procedure with some of the conceptual knowledge on which it is based, then the procedure becomes part of a larger network, closely related to conceptual knowledge. ... Procedures connected to networks gain access to all information in the network (Hiebert & Carpenter, 1992, p. 78).

When we combine Simon's teaching model with Driver and Oldham's teaching model, conceptual change process occurs in the learning trajectory. The big ideas that children come up with during the first phase of the trajectory take their original forms through the conceptual change process.

## Learning Geometry

> A small child draws first, then decides what it is that he has drawn; at a slightly older age, he names his drawing when it is half done; and finally he decides beforehand what he will draw (Vygotsky, 1962, p. 17).

Vygotsky gives a brief view on a child's developmental sequence, the evidence of which is based on drawing and naming the drawing. Attributing Piaget and Inhelder (1967), Clements and Battista (1992) suggest that drawing is an act of representation, not of perception. Piaget and Inhelder also claim that topological features develop first. Arf (1960) states that geometric understanding develops through the transition from the concrete level of understanding to the abstract level of understanding the geometric shapes. Children conceptualize and abstract geometric shapes through inquiry during the transition. Initial facts of mathematics, numbers and geometric shapes such as point, line, plane, etc. become abstract conceptions relieving their meanings that we attribute to our goods (Arf, 1960; Van de Walle, 1998). "For example, 3 apples, 3 trees, 3 pencils, and 3 birds at last create the concept of 3" (p. 34). Hiebert and Carpenter (1992) connect the construction of this concept with the procedural knowledge, which becomes a part of the network.

NCTM (2000) has some expectations for geometry from students in grades 6-8 under four basic categories. Students in grades 6-8 should *analyze characteristics* and properties of shapes and understand relationships among two- and three-dimensional objects; understand relationships among parts (e.g., angles, sides, heights, etc.) of geometric objects; and make inductive and deductive discussion on geometric relationships (i.e., congruence, similarity, etc.). Students in grades 6-8 should also *specify locations* and use the coordinate system to represent certain geometric shapes. *Transforming* objects and analyzing mathematical situations, as well as examining congruence and similarity using transformations, are among the expectations of NCTM from students in grades 6-8. Students furthermore are expected to provide reasoning for the geometric model to solve problems, draw a geometric shape with given properties, represent two- and three- dimensional objects using *visualization*, and represent algebraic relationships using geometric modeling.

### *Spatial Understanding*

There is a common point that many scholars agree on about geometry and spatial understanding: Geometry is an atmosphere that encircles individuals, and they must learn to live in it and break the *egg* to see other spaces (Cathcart, Pothier, Vance, & Bezuk, 2001; Clements & Battista, 1992; Grayson & Reynolds, 1999; NCTM, 1989, 1991; Van de Walle, 1998; Yakimanskaya, 1991). "Spatial sense is an intuitive feel for one's surroundings and the objects in them" (National Council of Teachers of Mathematics, 1989, p. 49). When we look at geometry from this perspective, we can see the beauty of our environment.

On the other hand, spatial thinking is a form of mental activity that makes it possible to have mental images of mathematical patterns and relationships. For example, it is possible to have a mental map of a city (Grayson & Reynolds, 1999; Yakimanskaya, 1991). Spatial image is a form of spatial thinking which is closely interconnected with other forms of thinking. This involves constructing and transforming mental images.

> The activity of image expression constitutes the basic mechanism of spatial thinking, and consists in the use and transformation of images; it is often a lengthy and repetitious process. This process incorporates images that arise in various, and therefore spatial thinking involves a constant recoding of images, a transition from spatial images of real objects to conventional graphic representation of these images or from three-dimensional to two-dimensional representations. (Yakimanskaya, 1991, p. 22).

One of the aspects of spatial thinking is that in the problem-solving skills, it appears as spatial reasoning which one has to create within their cognitive network. "Spatial reasoning consists of the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated" (Clements & Battista, 1992, p. 420). Thus, spatial thinking results in a transition from spatial images of real world objects to other types of representations of images such as graphics, and a reorganization of images. Mental images manipulated by (spatial) thinking lead students to be flexible in the problem-solving process. "A strong spatial sense allows students to formulate image-based solutions to mathematics problems" (Grayson & Reynolds, 1999, p. 375).

Understanding of shapes, what they look like, and what they are named help students to develop their spatial sense (Van de Walle, 1998). Geometric concepts such as symmetry, congruence, and similarity have a big effect on understanding the geometry around us. According to Owens (1990), spatial sense has two components: visualization and orientation. Spatial visualization requires mental picturing of objects under motion or other transformations in two- and three-dimensional space (Cathcart et al., 2001; Clements & Battista, 1992). Spatial orientation includes understanding the positions of objects from various perspectives (Clements & Battista, 1992). Imagining a figure requires spatial ability, which is transforming visual images and understanding abstract relationships between visual representations.

While Owens (1990) and Clements and Battista (1992) have characterized spatial thinking as a two-factor ability, Piaget conceptualized spatial thinking as a developmental sequence where children develop topological thinking first in the sequence of spatial thinking, and then progress to spatial thinking in Euclidean space, and finally explore abstract objects with Euclidean characteristics. In support of Owens (1990) and Clements and Battista (1992), Lohman (1979) conceptualized a three-category spatial thinking: spatial relations, that is, mental rotation of a visual stimulus; spatial orientation, that is, how an object appears respect to one's perspective; and visualization, that is, the redesign of spatial information, such as the mental picturing of geometric figures (Rosser, 1980).

*Knowledge Development*

There are some developmental issues relevant to learning. Although many theorists agree on the definition of development presented earlier, these issues recently have been raised in terms of how development occurs, what it depends on, if it happens continuously via small changes or happens suddenly, what roles children have in their development, and if development consists of changes in cognitive structures or processes (Schunk, 2000). One of the debates is between behavioral and cognitive theorists; that is, "Does development occur continuously via small changes or do sudden, abrupt changes occur?" (p. 222). Behavioral theorists claim that as behaviors develop continuously, they shape new ones. By contrast, Piaget's theory explains a discontinuity of development. There may be sudden changes from one mode of thinking to another. Schunk (2000) attributes to Piaget that cognitive development passes through a fixed sequence, and this development depends on biological maturation, experience with physical environment, experience with social environment, and equilibration (adaptation). According to Piaget, for cognitive development, individual receives conflicting environmental input (Schunk, 2000). How children view the world forms the stages of thought (see Table 2.1).

The stages that Piaget characterized have some assumptions:

1. The stages are discrete, qualitatively different, and separate. Continuous merging is not necessary for the progression between stages.
2. Prior development affects the next development of cognitive structures.
3. Cognitive development in a particular stage varies from person to person, but the order remains constant. In other words, one may reach, for example, to concrete operational stage at the age of fifteen.

Table 2.1 Piaget's stages of cognitive development and their characteristics (adapted from Schunk, 2000).

| In this stage: | individual |
|---|---|
| Sensorimotor (birth to 1.5-2 years) | acts spontaneously, has internal motivation; constructs knowledge at a primitive level; and changes rapidly in the period cognitively. |
| Preoperational (2 to 7 years) | imagines the future and reflects the past, but is unable to think in more than one dimension at a time and demonstrates *irreversibility* (e.g., the box flattened cannot again be made into a box); has difficulty distinguishing fantasy from reality. |
| Concrete operational (7 to 11 years) | shows remarkable cognitive growth; gets language and basic skills dramatically; begins to show some abstract thinking; acquires reversibility in thinking along with classification and seriation – acquisition of mathematical skills. |
| Formal operational (11 to adult) | develops operational thought; is able to think about hypothetical situations; improves reasoning capabilities; can think about multiple dimensions and abstract properties; and shows idealistic thinking. |

Piaget and Inhelder's (1967) theory describes children's conception of space. There are two major themes of this theory: First, the child acquires representations of space through the progressive organization of his/her motor and internalized actions (Clements & Battista, 1992). Thus, the development of representation of space is intertwined with the individual's previous manipulation activity. Second, geometric ideas progress through a logical, rather than chronological, order; that is, "initially topological relations (e.g., connectedness, enclosure, and continuity) are constructed, and later projective (rectilinearity) and Euclidean (angularity, parallelism, and distance) relations" (Clements & Battista, 1992, p. 422).

Piaget and Inhelder also discuss that perceptual space is constructed in the sensorimotor period. Initial perception of geometric shapes is constructed during the first stage of development through explorations by touching some parts of a shape (Clements & Battista, 1992). The experiences of the environment help children to abstract geometric shapes by conceptualizing.

Spatial thinking develops differently in each individual. One of the characteristics of Piaget's stages of cognitive development is that "although the order of structure development is invariant, the age at which one may be in a particular stage will vary from person to person" (Schunk, 2000, p. 234). The ability to construct and manipulate and to transform gives students the opportunity to learn about position and movement in space and the properties of shapes. Students discover relationships and develop spatial sense by using this ability depending upon the level they have achieved. "Spatial sense is clearly enhanced by an understanding of shapes, what they look like, and even what they are named" (Van de Walle, 1998, p. 348). Students' previous experience in mathematics affects their improvement in spatial thinking. Interacting with peers, students will get the idea that there may be more than one way to solve a problem, and they will be encouraged to use their own ways that make sense to them instead of relying on a teacher to show the way. "Due to contact with others, the child becomes ever more aware of his or her own thought, ..." (Clements & Battista, 1992, p. 440).

Children's understanding of geometric shapes involves touching one side of an object and understand the part, then touching other sides to grasp what they look like, then combining these parts, and finally conceptualizing the shape (Clements & Battista, 1992). The position of an object in space may be related with another one, that is, other objects around an object help the individual to locate it in his/her mind. Students use the same mental activity when they develop a spatial sense of geometric shapes, especially the features of a shape and relationships among the parts within an object. This is important for improving spatial thinking in geometry. Clements and Battista (1992) emphasize the importance of spatial thinking in geometry. Furthermore, they refer to Soviet researcher Yakimanskaya to give the connection between visualization and geometric knowledge and individual concepts: "... understanding the concept of rectangle and its properties requires that students analyze the spatial relationships of the sides of a rectangle-that is, understand 'opposite' sides and distinguish them from 'adjacent' sides" (p. 443).

*The van Hieles: Levels of Geometric Thinking and Phases of Instruction*

Dutch educators Diana van Hiele-Geldorf and Pierre van Hiele have formulated a theory of five levels of geometric thinking, each of which has various "phases" in proceeding from one level to the next. This model identifies the geometric thinking process from a visual level through more sophisticated levels of description, analysis, abstraction, and proof (Clements & Battista, 1992; NCTM, 1988). This philosophy maintains that learners progress sequentially from visualization toward the rigor level that is comparison of axiomatic systems of geometry. According to the van Hieles, the learner, facilitated by appropriate instructions, moves sequentially through five levels – visualization, analysis, abstraction, deduction, and rigor – where one level of thinking cannot be achieved without having passed through the previous levels for every geometric idea learned.

Before giving the levels, I will mention the instructional phases of learning that lead a student from one thought level to the next, where the next level starts with the product of the previous one (Crowley, 1987; Van de Walle, 1998). Learning and teaching sides of the model are important issues through the phases that suggest an interaction between students and the teacher.

*Phase 1: Information/Inquiry.* At this initial stage the teacher and students engage in conversation and activity about the content. Through this conversation students get familiar with the objects, and have vocabulary for that level. Materials related to the level are presented to students as a prerequisite knowledge to the learning period between the levels. Teacher learns about the students' knowledge (e.g., how they interpret the new information about the content) by raising questions.

*Phase 2: Guided Orientation.* Students explore the content through the materials provided carefully by the teacher. Students encounter activities that bring out different relations of the network that is to be formed (e.g., folding, measuring, and looking for symmetry). For example, students at this phase between level 1 (visualization) and level 2 (analysis) can discuss the properties of shapes, for instance, revealed by reflection.

*Phase 3: Explication.* Building on their previous experiences, students express and exchange their emerging views about the structures that have been observed (e.g., express ideas about the properties of a figure). Students become aware of the relations, try to explain them in their own words, and learn technical language for that subject matter. The teacher's role, which Battista and Clements (1992) attributed to the van Hieles, is to bring the language of geometric objects, ideas, relations, and patterns to an explicit level for the students. Once they have had mathematical terminology, they can transfer their own discussion to the more formal situation.

*Phase 4: Free Orientation.* Students encounter more complex tasks that can be completed in more than one way to learn to find their own way within the network of relations (e.g., knowing properties of one kind of shape, investigating these properties for a new shape) (Clements & Battista, 1992; Teppo, 1991). The role of the teacher is to provide relevant materials and geometric problems that help children gain experience in explicit relations among the objects of the structures being studied. Students are given more open-ended activities that can be approached by several different kinds of solutions.

*Phase 5: Integration.* Students summarize all they have internalized about the content, and then integrate their knowledge into the newly formed network of relations (e.g., properties of a figure can be summarized). The role of the teacher is to encourage students to reflect on their conceptions, and to use them in a new situation. The completion of Phase 5 allows the students to attain a new level of thinking for the studied topic (Clements & Battista, 1992).

*Levels of Thought and Their Characteristics*

The levels through which an individual passes completing the phases above describe "how the individual thinks at each level, and what kinds of geometric ideas the individual thinks about rather than how much knowledge one has" and the way in which the understanding of a new topic may develop (Van de Walle, 1998, p. 346). As people progress through qualitatively different levels of understanding, with the activities facilitated by the teacher in the learning environment, the context of their geometric thinking shows some characteristic differences, but "the product of one level becomes the object of the next level" (Clements & Battista, 1992; Van de Walle, 1998, p. 347). This relationship can be seen in Figure 2.2. The levels of geometric thinking have a strong hierarchy. One important aspect of the van Hiele levels is that an instruction presented at a higher level of thought cannot be understood by students at a lower level of thought. Indeed, Cathcart et al. (2001), Van de Walle (1998), and Teppo (1991) state when students and teachers *speak* at different levels, they *literally* do not understand one another.

To get rid of confusion about the levels, I will number the levels from 1 to 5, as seen in many articles and books.

*Level 1: Visualization.* Children at this level classify shapes according to their appearance without attention to their parts or properties. They have their own prototypes such as doors and balls. They reason by holistic resemblance of shapes. Therefore, "the objects of thought at *level 1* are shapes and what they 'look like'" (Van de Walle, 1998, p. 346, italics are modified to this study). Since in their thinking the properties of shapes are not explicit yet, they classify shapes as "alike." For example, "a 'thin' rectangle may not be classified as a rectangle because it is 'too thin' relative to the student's rectangular prototype of a door" (Fox, 2000). Reasoning at this level is likely intuitive and recognition of an object relies heavily on visual perception. "There is no *why,* one just sees it" (van Hiele, 1986, p. 83). Consequently, "the products of thought at this level are classes or groupings of shapes that seem to be 'alike'" (Van de Walle, 1998, p. 346).

*Level 2: Analysis/Description.* Students at this level can easily classify or group, name, and compare objects in terms of their appearance – the achievement of level 1. Students analyze figures and shapes on the basis of their properties and relationships among parts/components. One at this level knows that opposite angles of a parallelogram are congruent or that an isosceles triangle has the property of having two equal sides and two equal angles. At this level students begin to discover that collection of properties goes with classes of figures (Clements & Battista, 1992; Van de Walle, 1998). Through observation and experimentation students start to realize the characteristics of shapes, and to generalize the class of figures, but not relations between properties or interrelation between figures. Children at this level rely heavily on their beliefs, not on assumptions; in other words, as

Clements and Battista (1992, p. 439) cited from Piaget, "the child cannot reason from promises without believing in them." The main objectives of this learning period between level 2 and level 3 are networks of relationships and the ordering of properties rules of geometric shapes. "The product of this level is establishment of relationships between and the ordering of properties and classes of figures" (p. 427).

*Level 3: Abstraction/Informal Deduction.* Students can develop interrelationships of previously discovered properties within figures (in an isosceles triangle, having two equal sides necessitates the angles that the two sides make with the basis being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle) (Sharp & Hoiberg, 2001; Van de Walle, 1998). They start to give informal deduction when they interrelate the properties and the classes (ordering the properties). They are able to engage in "if-then" reasoning (if it is a square, it must be a rectangle), that enables them to classify shapes with minimum characteristics (rectangles are parallelograms with a right angle) (Van de Walle, 1998). On the other hand, "the students still do not understand that logical deduction is the method for establishing geometric truths" (Clements & Battista, 1992, p. 427). The objects of this level are properties of classes of figures; therefore, the products of thought at level 3 are "the reorganization of ideas achieved by interrelating properties of figures and classes of figures" and "relationships among properties of geometric objects" (Clements & Battista, 1992, p. 427; Van de Walle, 1998, p. 346).

*Level 4: Formal Deduction.* At this level, students examine axioms, theorems, definitions, and postulates by using the previously produced conjectures concerning relationships among properties (Van de Walle 1998). Students are able to prove theorems deductively and establish interrelationships among these theorems, axioms, definitions, and postulates. Students begin to appreciate the need to prove a logical axiomatic system and show ability to prove in more than one way. The objects of thought at this level are relationships among properties of classes of figures; thus, the products of this level are "the establishment of second-order relationships – relationships of relationships – expressed in terms of logical chains within a geometric system" and "deductive axiomatic systems of geometry" (Clements & Battista, 1992, p. 428; Van de Walle, 1998, p. 347).

*Level 5: Rigor.* Theorems in different axiomatic systems can be compared. Non-Euclidean geometry can be studied and geometry is seen in the abstract. "The products of thought at level 5 are comparisons and contrasts among different axiomatic systems of geometry" (Van de Walle, 1998, p. 347).

Many scholars agree that the majority of high school geometry courses are taught at level 4 (Formal Deduction). Burger and Shaughnessy (1986), Suydam, (1985), and Usiskin (1982) have shown that almost 40% of high school graduates are at or below level 3 (Abstraction/Informal deduction) (Clements & Battista, 1992). This mismatch indicates that students may not benefit from the learning periods between levels, as it should be. Then the question becomes: Whose fault is it: teachers', textbooks', or students'?

*Characteristics of van Hiele Levels*

There is a hierarchical relationship between levels, that is, to move from one level to the next, one needs to achieve some abilities at that level so that he/she can use this ability in the next level of activities. This relationship, the "object-product" relationship as Van de Walle (1998) called it, is illustrated in Figure 2.2. The transition between consecutive levels has a five-stage instruction to help children move to the next level.

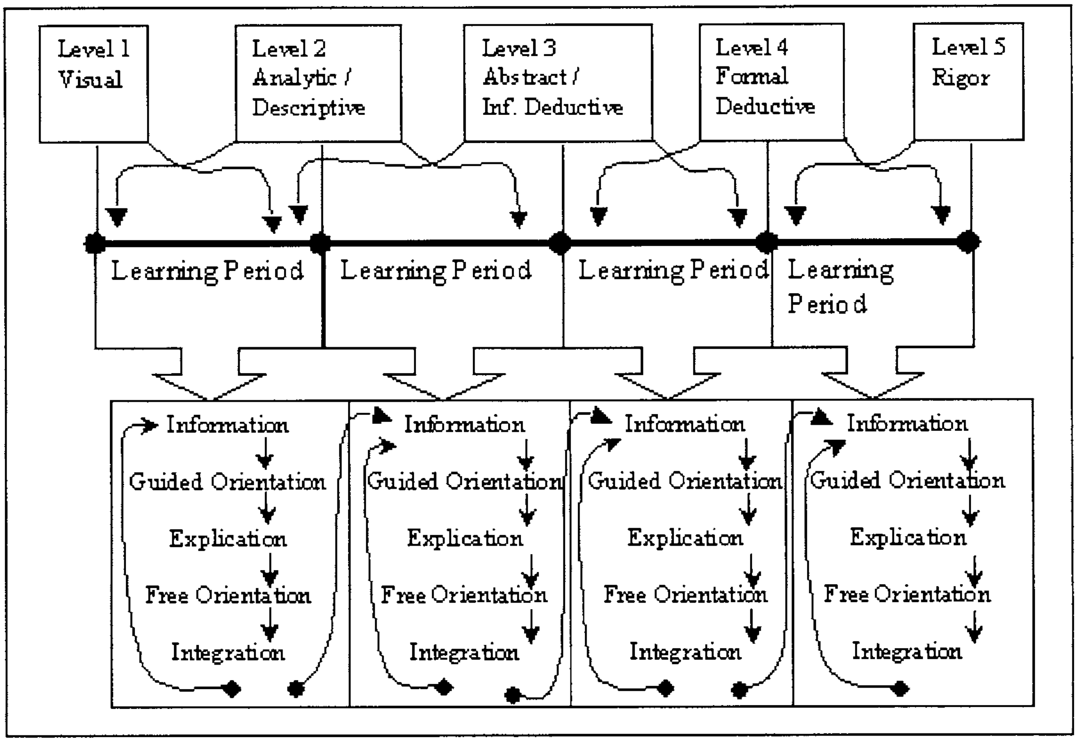



Figure 2.2. Transition from one level to the next; that happens in the learning period with the phases of instruction (adapted from Sang, 1999; and Van de Walle, 1998).

1. The levels are sequential; that is, children must pass through the levels as they experience appropriate geometric thinking for that level.
2. Progress from one level to the next is dependent on the experience that students have had, and the instruction, rather than age or maturity. One can be at any level given a geometric concept.
3. Each level has its own terminology; therefore, a geometric concept, say quadrilaterals, can be perceived in different way "at" different levels. The language used in instruction must be suitable for students' level of thought. For example, one student can memorize that a rhombus is a type of parallelogram without internalizing the relationship behind that statement.

4. Since the levels are sequential, concepts implicitly understood at one level become explicit at the next level (Clements & Battista, 1992).
5. Movement between levels is made possible by instruction.

However, it is appropriate (and likely) that students will revert back to a previous level (for a moment) when confronted with a new task. Also, there is considerable folding back and forth between adjacent levels when the student needs it. In other words, adjacent levels overlap each other, and how much wide this overlap will depend upon the learner's achievement during the learning period. Through the learning periods of consecutive levels, the learner shows the traits of both levels. The later an individual reaches the next level, the more he/she is dependent on the previous level.

## Implementations of Constructivism in Geometry

From a constructivist standpoint, the student is not a passive consumer of knowledge exposed by the teacher; rather he/she is an active learner who constructs his/her own understanding of content by interacting with what he/she currently knows in the learning environment (Hewson, 1992; Posner et al., 1982; Schunk, 2000; Scott, 2001; Simon, 1995). Consequently, in instruction based on the constructivist learning approach, the student is the center of learning and the teacher is a facilitator who creates an appropriate learning environment, rather than the student being a passive receiver and the teacher an active donor. Constructivist views of learning have offered theoretical frameworks for mathematics educators and for teachers to understand their students. However, with this learning theory, most teachers are challenged to develop new teaching models. To contribute to the development of mathematics pedagogy, Simon (1995) clarifies the misconception or misinterpretation about constructivism, saying:

> It is overly simplistic and not useful to connect constructivism to teaching with the romantic notion, "Leave students alone and they will construct mathematical understandings." Likewise, "Put students in groups and let them communicate as they solve problems," is not much more helpful (pp. 117, 118).

Developing geometric understanding and spatial reasoning must be achieved by the learner himself/herself. In fact, if we consider the constructivist approach as a way of learning mathematics, students will construct the mathematical relationships and build a network of mathematical knowledge as they interact with each other in small groups as well as with the teacher. Instead of directly introducing mathematical concepts to pupils ("finished products of mathematical activity"), students should have the opportunity to reconstruct the content through inquiry (De Villiers, 1998). From a constructivist standpoint, students should be actively involved in the learning process of defining geometric concepts rather than learning definitions (Cobb et al., 1992; De Villiers, 1998). The role of a teacher is to be a facilitator who fosters conceptual knowledge, not a transmitter who is the only authority of knowledge (Henriques, 1997; Matthews, 1994; Schunk, 2000; Scott, 2001; Shymansky, 1994; Sivertsen, 1993; Yore, 2001).

In a similar vein, Clements and Battista (1992) emphasize that "the van Hiele theory, though, does not support an 'absorption theory' model of learning and teaching," although the teacher has a crucial role in developing the geometric thinking by providing appropriate guidance (p. 430). Achieving a higher level occurs not via a teacher's telling, but via students' own readiness for the next level as a means of a proper selection of activities. Likewise, in the constructivist theory, students have the control to determine what they will interpret from the instructional representation, not the teacher (Driver & Oldham, 1986;

Posner et al., 1982; Simon, 1995). Moreover, attributing to Brousseau (1981), Simon (1995) mentions the responsibilities of the teacher for planning of instruction, which has an important role in students' learning. However, we should also be careful how much power the teacher has on students' learning of mathematics. Radical constructivist Ernst von Glasersfeld (1999) highlighted this issue with the answer of the question asked by Carla DeLancy via electronic communication saying, "...In my view, teachers can only SHOW how knowledge could be constructed, they can never transfer what they happen to know (Ernst von Glasersfeld's Answers, April 1999, http://www.oikos.org/vonanswerapril.htm). Simon (1995) furthermore emphasizes that for developing mathematical understanding, students should have the freedom to use their previous knowledge of the context to make comments on a situation. "If the situation *leads* the students to a particular response, no real learning of the mathematical ideas underlying that response takes place" (p. 119). Yet, he quoted Brousseau (1987, p. 8) to focus on the importance of instruction in mathematics education: "If the teacher has no intention, no plan, no problem or well-developed situation, the child will not do and will not learn anything" (p. 119). From the realm of these discussions, we can conclude that students must learn how to develop their discussion on a particular situation rather than learning how to answer the teacher's questions.

In developing spatial thinking, Wheatley and Cobb (1990) also claim that children interpret spatial patterns how they understand them based upon their experiences, cognitive structures, and social communications. In addition, Hand (1996) insists on the role of the teacher is to allow students to build *contextualized* content knowledge rather than to transmit *decontextualized* knowledge. He further suggests that students learn facts and algorithms by heart, getting decontextualized knowledge. As a result, students give arguments in proving geometric axioms attributing to their classroom activity; "this is true because it is a theorem or procedure I learned in class" (Clements & Battista, 1992, p. 442).

## Affective Domain and Mathematics Achievement

> "How one thinks about mathematics education, how one defines what it means for students to know and do mathematics in school, is, in turn, affected by one's views about the nature of mathematics, one's underlying epistemological perspective, and one's educational goals" (Teppo, 1998, pp. 7, 8).

Many scholars (Cobb, Yackel, & Wood, 1989; Fennema & Sherman, 1976; Grouws & Cramer, 1989; Hart, 1989; Lester, Garofalo, & Kroll, 1989; McLeod, 1989; Picker & Berry, 2001; Schoenfeld, 1985; Siver, 1985; Teppo, 1998) have studied affective factors in mathematics education. These affective domains are beliefs, attitudes, and emotions, which have crucial effects on mathematical problem solving. McLeod (1989) expresses that beliefs, attitudes, and emotions may change depending upon students' performance in solving problems. Students have an attitude toward mathematics, mathematicians, and mathematics teachers. These students are affected by the society's views of mathematics in early ages (Mandler, 1989; Picker & Berry, 2001). As a result, mathematics may become their nightmare among school subjects. One's first impression toward someone leads him/her to determine how he/she behaves in future. The same things happen in mathematics due to students' perceptions of mathematics. The emotions that children have about mathematics, especially in early grades, form their belief system of mathematics and their attitudes toward mathematics. Mandler (1989) emphasizes that the emotions we have about mathematics affect our learning of it, which is related to understanding mathematics. Mcleod (1991) mentions two dimensions of affective domain: intensity and stability. She compares emotions, attitudes, and beliefs across these two dimensions, and states:

> ... emotions are intense but unstable (in the sense that they do not last long); attitudes are less intense than emotions but more stable; and beliefs are less intense and more stable than attitudes (attitudes can be changed, but beliefs are difficult to change). (Mcleod, S. 1991, http://jac.gsu.edu/jac/11.1/Articles/6.htm).

Taking this into account, one's emotional reaction to mathematics may shape the attitude (positive or negative) toward mathematics, and this positive or negative attitude may be inferred from one's belief about mathematics. Paying attention to affective domains having place in one's belief system, Hart (1989) makes a distinction between interest in belief systems and interest in attitudes. "Interest in beliefs and belief systems has come mainly from cognitive psychology, whereas most of the interest in attitudes and affect has come from social psychology" (p. 41). At this point Hart attributes to Schoenfeld (1985) the description from the mathematical side, saying that belief systems consist of conceptions about the nature of mathematics, especially "the constitution of mathematical arguments" (p. 41). There are many conditions in the environment that influence one's belief system about mathematics. Teachers are one of the major factors that affect students' beliefs about mathematics. Teachers raise children to use mathematics in a way in which they believe. Students do not believe that they will use the knowledge they learn in the classroom (Hart, 1989). Such a belief causes them to accept what they are taught without attempting to understand it. Schoenfeld (1985) argues that this kind of learning hides children's ability to learn. Stating the importance of the belief about mathematics, Hart (1989) expresses that students' mathematical understanding may be limited by their teachers' beliefs about the nature of mathematics, which may be different than what they do in their mathematics classes. Considering problem solving issues, McLeod (1989) gives two major classes of beliefs that have an influence on mathematics learners: "beliefs about mathematics made by students; and self-belief about themselves" (p. 247). Furthermore, "beliefs about mathematics play a central role in problem solving by experts as well as novices and by teachers as well as students" (p. 248).

In a learning environment, positive attitudes have essential roles in ascertaining the ability of students to learn. Some educators (Cobb, Yackel, & Wood, 1989; Grouws & Cramer, 1989; Hart, 1989; Mandler, 1989) say that the ability of children to learn is affected by their positive attitudes. Others state that positive attitudes are an educational result, regardless of students' attitudes. There are other views about attitudes toward mathematics, focusing on a positive correlation between affective variables and the understanding of mathematics (Fennema & Sherman, 1976; McLeod, 1989; Schoenfeld, 1985; Siver, 1985). In other words, positive attitudes lead to more achievement in mathematics, and, in turn, improved mathematical understanding results in more positive attitudes. Hart (1989) disputes that these affective variables may change across different branches of mathematics. "One student may be more confident about algebra than geometry and more confident when working alone than when explaining a concept to a class of peers" (p. 39).

# METHOD

## Context

I sought a participant for this study who was a sixth grade student in Ames, Iowa. The principal reason for choosing the participant out of my contacts was to select a student who as at secondary school level, who had a different mathematical perspective, and who would voluntarily participate in the study.

According to the consent process that I followed, I will not reveal the name of the participant. For this reason, I decided to use "ogrenci," which means "student" in Turkish. In this way, I not only obeyed the agreement but I avoided confusing readers with the term "student" that was often mentioned in the study.

Ogrenci was a sixth grade student at a public school in Ames, Iowa, during the 2002-2003 school year. The youngest child of the family, Ogrenci was originally from Saudi Arabia and came to Ames six years ago, when he started first grade. He said that he never liked mathematics, even hated it, because of the mathematics problems he had as either homework or at school. His mother was a doctoral student in the science education department at Iowa State University, where I met her, and his father was also a doctoral student in the microbiology department at the same university. Since the mother was in education, she was willing to help her children's schoolwork from an educator's perspective. She always mentioned her son's attitude toward mathematics, and asked me how she could change his negative attitude. While I was thinking what to do for my thesis, these conversations came to my mind and I asked her whether her son would want to participate in this kind of study. At last, we went through the consent form and we agreed to do this study.

## Data Collection Instruments

My role in this study was: 1) as an observer, to analyze the participant's behavior while doing mathematics, and 2) as an interviewer, to examine his geometric thinking by asking questions. I also had the role of instructor during the sessions. Audiotape, field notes, and observations were used as the instruments of data collection. Due to the nature of qualitative research and the scheduling of learning periods, I prepared study topics before the week during which each session (here, the term "session" refers to each meeting time with the participant) was held (Gay & Airasian, 2000).

## Procedures

Having received approval for the research from the ISU Human Subjects Research Office; I set up meeting times and located an office area that Ogrenci and I agreed was acceptable. We got together every Sunday from 12/02/2002 until 3/9/2003, except during school breaks and some other occasional times when it was impossible to meet (e.g., special family meetings, illness, or heavy snow). We met nine times. I also met his teacher at the school to get information about the curriculum that he followed in mathematics.

From the very beginning, I started to audiotape the whole session when we met to study. Besides audio taping each session, I also took notes and observed him during the studies. The data were analyzed by revisiting my field notes and by listening to the audiotapes. After each session, audiotapes were transcribed for analysis, and a reflective notebook was kept following the sessions.

In this study, I assumed both the roles of teacher and researcher. This put me in a position where I was able to examine my interaction with Ogrenci through my theoretical framework regarding learning and teaching. Particularly, this study focuses on the participant's responses to the questions and activities about the specific geometric context, e.g., two-dimensional shapes such as polygons, relationships among polygons, and spatial relationships among geometric shapes. Before each session started, I went to the participant's house to pick him up and brought him to the office where we had our studies. Since he was not favorably disposed towards mathematics, I tried to reduce his stress by talking, during the journey to the office, about daily issues, his weekdays, and his attitude toward mathematics. Because in certain contexts, an individual might choose not to fully reveal his/her knowledge about a given topic (Simon, 1995) – and this certainly was valid for my participant – the mathematical content of the sessions began with various informal geometric explorations (i.e., tangram problems). I should note that in the very early sessions, I had prepared some questions to assess his prior knowledge about the topic we would study, but this did not work. Ogrenci did not want to answer all the questions or gave short answers that did not help to understand his knowledge of the content. Later, I still prepared those questions, but I posed them through the activities instead of giving them at the beginning of the sessions. Each starter-problem's context involved the structures of the previous content, so he could manage to engage in the mathematics activity we were doing. The first session did not start by addressing progress through the van Hiele levels. Rather, we did some fraction problems, mostly based on computation, and then from here, we further went to representations of fractions with manipulatives. These representations are based upon geometric shapes.

The instruction was based on the participant's thinking aloud, and discussing with me, rather than on my lecturing. Seeing that he might accept me as an authority of mathematics knowledge, I stayed away from revealing my thoughts fully. I wanted him to make sense of the mathematics on his own, and become confident in his own ability to do mathematics. Ongoing assessment enabled me to evaluate his understanding the topic, his connecting with his current knowledge, and his making sense of new information. Besides this assessment, audiotaping the sessions was very helpful for me to decide what he had learned from that session and what I needed to prepare for the next study.

## Design

To better understand my student's thinking in this learning environment, a case study design was most appropriate. A case study is a methodology in which researchers focus on the characteristics of a single person or phenomenon (Gay & Airasian, 2000). I used this approach to guide my implementation of the study and to evaluate the data gathered from the study.

## Data

The data (audio tape transcripts, teacher diary, written student work, and other teacher notes) were analyzed to determine the van Hiele level of the student's thinking for the task in question. The analysis takes into account both the learning periods between levels as well as the levels. An example of a hypothetical student's response for each level was described more fully in Chapter 2. However, below is an example of how I used the van Hiele levels to analyze the data.

## *Analysis of Data*

### *Level 1: Visual Level*

The main characteristics of level 1 are classifying shapes according to their resemblance, reasoning intuitively, and relying on visual perception. "I don't know *why* but I just see it" (van Hiele, 1986, p. 83).

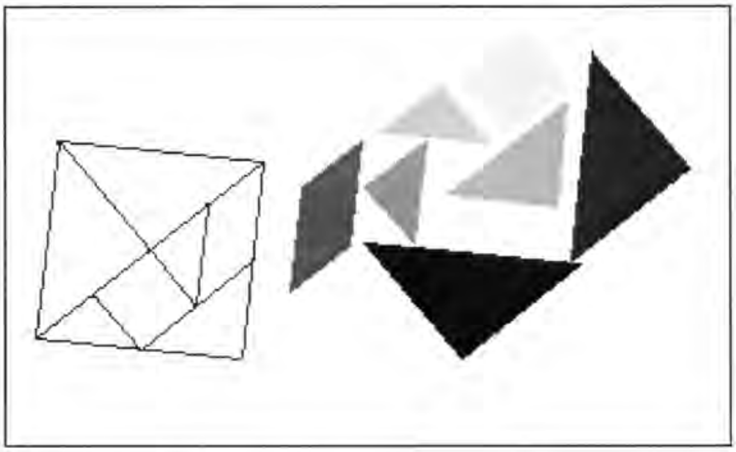
*Learning period.* Students move from level 1 to level 2 by experiencing classes of visual objects, which begin to be associated with their properties.

To understand Ogrenci's readiness for the activities, I introduced the tangram, which consisted of seven common geometric shapes such as square, triangle, and parallelogram during Phase 1: Information/Inquiry, in which materials related to the visual level (level 1) are presented to students. Students can learn geometric shapes and language. They also can reproduce a given shape (see Figure 3.1a).

Ogrenci reorganized the tangram pieces using his visual perspective. This activity was based on his spatial sense and his thinking of the relationships between tangram parts, which would help him to transit to level 2. Like a level 1 thinker, Ogrenci's performance did not indicate explicit thinking about the properties of the shapes. According to the descriptors for visual-level thinkers given by Fuys et al. (1988), Ogrenci identified the parts of the figure but did not analyze the figure in terms of these parts, or think of properties as characterizing a class of figures. I was expecting him to explain as "since the square has right angles, I cannot fit it to the base-corners of triangle, for the base corners I should use triangles or parallelogram." Rather, he used a "trial and error" strategy in the tangram puzzle. On the one hand, he was capable of recognizing shapes and separating the figures according to their characteristics. On the other hand, he did not use this prior knowledge, which is very important in the learning environment, while reorganizing the shapes.

Ogrenci was also given sorting-shapes tasks that required finding out shapes that were alike in some way (see Figure 3.1b). Ogrenci was able to sort out the shapes according to the number of sides. When he started to classify the shapes according to the number of sides, he was able to make piles of polygons (three-sided, four-sided, etc.) but not of the curvilinear shapes. In other words, he had difficulties placing the curvilinear shapes, especially the semi-curvilinears. He easily identified ellipsoid and circle but he could not put the shapes  and  into some categories. For the first semi-curvilinear shape, he said that it could be a rectangle except that it had curved sides. He immediately added that he was not quite sure about this shape by giving his reasoning as "if we are counting the number of corners, this shape [for the first semi-curvilinear] can go into four sided shapes, or if we are counting the number of sides, still this has four sides, but I am not pretty sure about other shape." In this particular task, he was able to make connection between his experience with geometric shapes and the new situation. This connection was made mostly based upon memorizing the relationships that had been instructed previously. This issue would be revisited in the subsequent phases of instruction.

*Further efforts for conceptual growth during the visual level.* What follows is a description of how I used the data to guide my efforts to move Ogrenci toward conceptual growth. In the previous example of data analysis, I explained how Ogrenci demonstrated that he was at level 1 for the ideas of geometric shapes. So, in typical case-study fashion, the procedures guided the data analysis and the data analysis effected my implementation of the procedures.



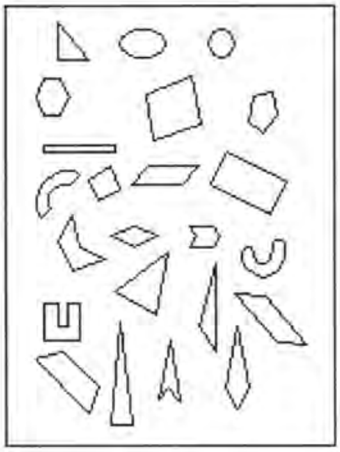
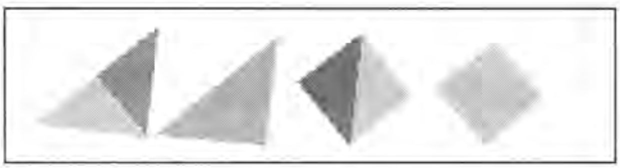


Figure 3.1a: Reproduce given a shape (Tangram). Figure 3.1b: Sorting-shapes-task.

To move Ogrenci toward analysis-type thinking, I moved to Phase 2 of the van Hiele theory of instruction. During Phase 2: Guided Orientation, students explore the content through the materials provided by the teacher. By folding, measuring, and looking for symmetry, they can construct a parallelogram or a square fitting two triangles into them without giving properties, but using resemblance of shapes (see Figure 3.2a).



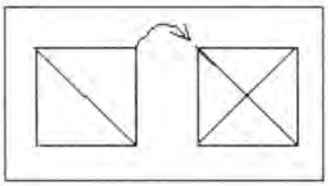


Figure 3.2a: Folding, symmetry. Figure 3.2b: Dividing square.

Although students may have an instrumental understanding of geometric shapes (e.g., square has four right angles, and opposite parallel sides), their understanding might be weak, not what is traditionally called relational understanding of geometric shapes (i.e., being an isosceles right triangle results in having two equal sides both adjacent to the right angle, and having two congruent interior angle of 45 degrees; or the perpendicular segment from the right corner to its opposite side creates another two triangles similar to the original one). One student may be taught that a triangle can be partitioned into two triangles by drawing a line from one corner to the opposite side; Ogrenci had the same type of instrumental understanding. Even though he made the bigger triangle using the little ones, he was not able to see the relationship (concept) that "two isosceles right triangles formed another isosceles right triangle in a certain way." Below is an example of how he had this practical learning and subsequently conceptual growth.

Recai: How do you know you can make two triangles from a square?

Ogrenci: I remember, one time our teacher told us that when we divide a square like this [he showed with his finger] we get two triangles (see Figure 3.2b).

Recai: Can you make it for me, please?

Ogrenci: Here it is [he drew a square, and divided into two triangles].

Recai: Is there any other way to divide the square into triangles?

Ogrenci: mm… yeah when we turn it, from this corner to the other one, over here… (see Figure 3.2b).

Recai: Go ahead and do it for me, please. [He drew another line from other corners]. Now, what do you have? [He had four triangles].

Ogrenci: b… but our teacher told us we could make two triangles…

Ogrenci was dissatisfied with this information that he had. In other words, the product of the information his teacher gave and his experience created a misconception. With this dissatisfaction he added new information to his existing concepts, and he assimilated the new idea. These narrative data indicate he experienced conceptual growth.

My next procedure was to use these data to move through Phase 3: Explication, in which students try to give properties of shapes based on previously observed objects, and explain them in their own words (e.g., triangle, square, parallelogram) (see Figure 3.3a).

Ogrenci had already known triangles, squares, and rectangles along with some of their properties. From Figure 3.1b, he had sorted out the shapes according to their number of sides. He knew that a rhombus is a parallelogram that has specific properties but he could not specify those properties. "A student can, for example, memorize that all squares are rectangles without having constructed that relationship" (Van de Walle, 1998, p. 347). On the other hand, he defined a parallelogram as a shape that has 3 or more sides (see Figure 3.3b). From his definition of parallelogram, a triangle, pentagon, and hexagon were all parallelograms.

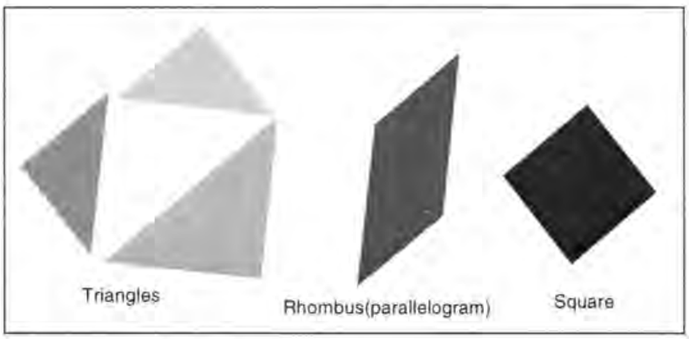



Figure 3.3a: Identify and name shapes.

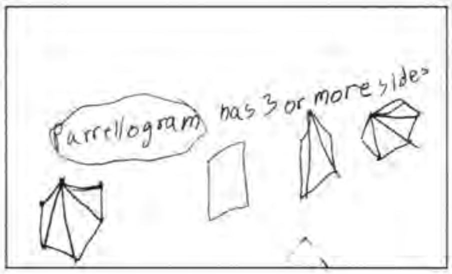



Figure 3.3b: Definition of parallelogram.

What follows is an episode that shows Ogrenci's classification of shapes according to his definition of parallelogram.

Recai: Can you show me which shapes are parallelogram? [He chose all *polygons*].

Recai: How a pentagon can be a parallelogram, can you explain this?

Ogrenci: These two sides [he showed two opposite sides of pentagon] are parallel to each other, that's why it's a parallelogram.

Recai: What about this shape [I showed the hexagon]?

Ogrenci: Hexagon? [Yeah] It is also a parallelogram because these two, these two and these two sides are parallel [showing opposite sides of hexagon].

As can be seen, *Ogrenci* had a problem about the "name" of the definition, otherwise, the definition itself was *almost* right. On the other hand, the actual problem was that he insisted on his definition. *"Parallelogram is a shape that has 3 or more sides, that's what I have learned."* This was the main learning problem that he had experienced. He brought this incorrect prior knowledge to the learning environment.

This information guided me to plan for Phase 4: Free Orientation, whereby students encounter more complex tasks that can be completed in more than one way, to learn to find their own way within the network of relations (e.g., using the same shapes, making other figures; and investigating the properties of the new figure) (see Figure 3.4).

After Ogrenci figured out the shape and reassembled the pieces, he was able to see that he could make the same figure by turning the two bigger triangles around the square. This is also related to spatial thinking; e.g., seeing that by rotating some parts of the tangram, the same shape can be produced. The difficulty that he had while doing this activity was to place the bigger triangles. The big hint I gave him was to advise him to think where first to place the two bigger triangles.

Having completed the activities related to the visual level of thinking during the first four phases of instruction, we were ready to proceed to Phase 5: Integration, in which students summarize all they have internalized about the content, and then invest their knowledge in a newly formed network of relations.

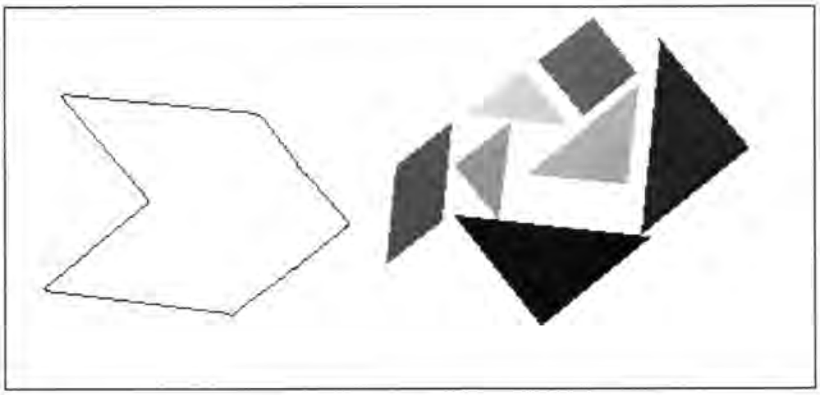


Figure 3.4: Given a different shape, finding the properties of the new shape.

Ogrenci summed up his observations and experiences about the tangram pieces and sorting-shape-task. For instance, "We have seven pieces that are different shapes, two little triangles, two bigger ones, one little square, one *rhombus,* and oh, one another triangle." He also gave the relationships between tangram pieces using his visual perspective.

This episode shows how I analyzed all of the data. I followed the van Hiele levels, which are hierarchical, and the phases of instruction, which have an important impact on the achievement of the levels, during the sessions. Each new session was planned based upon the previously analyzed session(s). Analyzing the data session by session helped me it easily, according to the van Hiele levels of geometric thought. I labeled my findings according to the traits of Ogrenci's performance (e.g., visual level descriptor: classifying shapes according to their appearance without thinking properties).

# RESULTS

After all the data were analyzed, I found 3 basic trends: (1) He had a strong, deeply ingrained, negative attitude toward mathematics, which was constant through the levels; (2) his reasoning skills could be classified according to the van Hiele levels of geometric thought. However, sometimes this was a difficult process because he often arrived at incorrect answers using reasoning skills that surpassed his basic geometry knowledge. For instance, sometimes, knowledge considered to be lower level (e.g., recognizing a parallelogram) was incorrect, but he could reason as if he functioned at a higher level. Other times, he appeared to possess higher-level knowledge but his reasoning suggested he was merely referring to information he learned from his regular classroom teacher; and (3) lastly, but most importantly, he showed different thinking levels across the geometric topics we worked on. I gave a comprehensive description of these trends below. However, an example of trend (1) is his wording that he never liked mathematics. For trend (2), Ogrenci seemed to have an impressive reasoning skill based on his own understanding of mathematical concepts (i.e., "my definition of parallelogram is a shape with three or more sides, so this figure is a parallelogram.") On the other hand, in a situation where he showed level 2 thinking characteristics, his reasoning was based on rote learning (e.g., rhombus is a specific *parallelogram* [polygon], that's what we have learned in the class.) At the visual level, as an illustration of trend (3), he could make a connection between collections of properties and classes of figures, which shows a characteristic of the abstract level (i.e., trapezoid is a parallelogram; other two sides don't have to be parallel because my definition of parallelogram is, at least two parallel sides; these two are parallel, the other two don't matter.)

## Attitude Toward Mathematics

After all sessions were done, we started to talk about his thought about this study, and how the activities were different than those he currently had completed at school. One important issue that led me think on was the instruction and the roles of the students and the teacher in the learning environment. He said that during our studies, he had to do more thinking (than in school) to get the basic ideas or the definitions. To my question "Which one did you find most difficult?" he responded as "in these activities I had to think mostly, and it is hard to do that. I prefer the teacher gives the definitions and the ways of problem solving, and then we learn them." We then had an argument whether this was more beneficial to learn mathematics. I was happy to hear from Ogrenci that having students engaged in the learning process would help them create their problem solving ways. He raised another crucial issue, the time problem, which as mathematics educators, we should think of. He said that working on the problems by himself took more time than learning from the teacher.

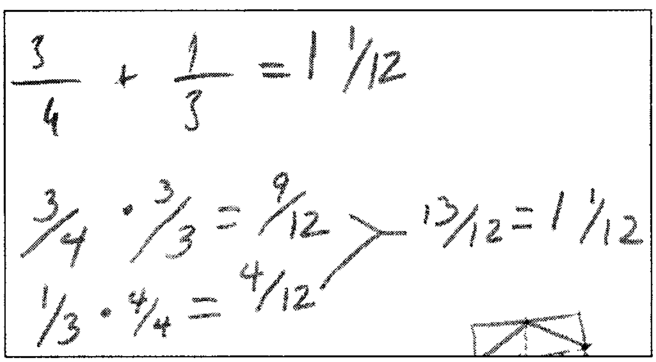



Figure 4.1: Addition of two fractions.

Now, I would like to go back to earlier sessions and give some of my observations about his emotions, attitudes, and beliefs of mathematics. When we first met to study, since I had known he did not like mathematics and mathematical activities, I tried not to ask complex mathematics questions, but some easy questions about the current topics that he had in the class (e.g., some fraction problems including their representations and addition and subtraction) (see Figure 4.1). He seemed to have, or he wanted me to think so, facility with mental computation. I concluded this from his behavior when he was solving the questions (i.e., without using pen for the addition of two tractions, "hmm ... this would be three-fourths, which is 75%").

Often, after being particularly successful with a given activity, or showing an interesting insight, he demonstrated a great deal of satisfaction. On these opportunities, he felt good at mathematics, but this had no effect on his attitudes or beliefs. These events just made him happy (change in his emotion) for the moment.

Early on, we agreed to have one-hour sessions. I noticed he would start to look at his watch towards the end of these early sessions. He wanted to finish up earlier and then get rid of mathematics immediately. However, when I had him engaged in the later activities, focused him on thinking about what he was doing, and used ongoing questions, he would not realize we had finished the hour. For instance, during the discussion of "polygons and parallelogram" (see Appendix), he was engaged in answering my questions and finding the correct definitions for the geometric shapes, which made him feel interested in the activities. At the end when I said, "Ok, I guess we are done for today, you have been showing great effort today", I assumed he felt happy when he disappointedly said, "Oh, ok, I think so."

He was happy when he took a teacher role. I sometimes pretended I knew less than him and let him explain to me the mathematics we studied. In this way students feel their ideas are valued and are listened to. As we were talking about his mathematics class at the school, he said that they had group discussions and then he added, "there is always someone responding to the teacher." In other words, all individuals are not required to be vocal in the class.

It was not my intention to try to affect his attitudes and beliefs about mathematics. However, as we were talking about his attitude toward mathematics, I tried to understand his emotions and attitude toward mathematics just observing him. His opinion was reflected in the view that doing mathematics does not mean liking mathematics or vice versa. He further said, "I know I am going to use mathematics in my future but this doesn't mean I need to like mathematics." He was aware of the usage of mathematics in a daily life, but he did not realize he was using mathematical thinking in his life. For instance, on the last day, we had a talk about the study, his school life, and mathematical thinking in his level (see Appendix). The building where we had sessions is like a maze; it is sometimes hard to find the way out. In order to have him to see he had at least mathematical thinking, I asked how he had gone out from the building. "I turned right and left, and find the way", he said. I pushed him to think how he used mathematics when he was going out by asking several questions. At first, he did not want to accept he had used mathematical thinking when going out. After some arguments, he realized that he had used "inverse relations."

## Reasoning Skills Through the Levels

## And

## Different Thinking Levels Across Various Topics

In this section, I will describe Ogrenci's reasoning skills through the levels (trend 2) and different thinking levels across various topics (trend 3). However, I will also relate some crucial points from the fraction episodes (see Appendix).

*Reasoning for Fractions*

There were two essential concerns regarding understanding of fractions: multiple representations of fractions and conversion of these representations to each other, and concept learning. He had no difficulty doing the four computations of fractions using numbers. Yet, he faced some discrepancies showing the processes with manipulatives. Although he was good at showing fraction amounts with circles, he encountered some troubles while he was using other representations such as rectangles, number lines, and sets. He could not transfer the representation of 2/3 with a circle to another representation, such as a number line model, or rectangle model. "I don't know how to divide rectangles, I only know to divide circles," he said. This reminded me a situation on the TV show "Sesame Street", which was mentioned in "Educational Psychology" book written by Goetz, Alexander, and Ash (1992).

> Grover monster is helping Ernie demonstrate counting to three. After Grover has successfully counted three blocks several times, Ernie puts away the blocks and brings out oranges. When asked to count the oranges, Grover sobbingly laments, "I cannot do it. I only know how to count blocks. I do not know how to count oranges!" (p. 60).

Goetz et al. emphasize that it is important that students internalize a mathematical concept abstracting the meaning of the concept. At this point, I had him to realize how he thought when doing with circles (see Appendix, Fraction episode 1). What he did cognitively was to transfer his existing idea of dividing to the new situation where he encountered a new phenomenon. This helped him to understand the idea of dividing an area model (Watanabe, 2002).

### *Reasoning Through the Levels*

In the rest of this section, I will give my findings about Ogrenci and his van Hiele levels.

*Recurring problems.* Throughout the study, we still had the recurring problem with Ogrenci's definitions of rhombus and parallelogram (see Appendix). Such a problem led me to think how hierarchical relationships among classes of shapes (e.g., quadrilaterals, rectangles, and squares) should be introduced to students. To better understand Ogrenci's problems, I reviewed the study of Kay (1987) about teaching quadrilaterals, rectangles, and squares conducted with first graders suggested that beginning with the more general case, and then proceeding to the more specific cases helped students embody the relationships among those shapes. In support of this study, de Villiers (1987) indicated that with a successful instruction, quadrilaterals could be taught first, and then how quadrilaterals could have special properties. The reason these two studies are relevant to Ogrenci's situation is because students in those studies experienced difficulty when the topics were not taught in a functional order that could help children's understanding of the geometric shapes.

*Attempts to address the problem with instruction.* These studies indicate the order in which topics should be studied. So, I took these suggestions into account, I first prepared a task about parallelograms, which is more general than rhombus. After Ogrenci gave a definition for "parallelogram" (see Figure 3.3b in Method section), we went on the sorting-shapes-task (see Figure 3.1b in Method section), and we then had a discussion on defining the parallelogram and placing his definition under the correct word (see Appendix).

*Reasoning with incorrect prior knowledge.* Although he showed visual thinking level traits on the task, he gave plausible claims for his definition of parallelogram, and could advocate his claims. His incorrect definition was still "a parallelogram is a shape with three or more sides." So, in a sense, he was reasoning correctly, but started with an incorrect premise that led him to an incorrect solution. Having given his definition, he could classify the shapes under his own definition. We continued to discuss why he had given this definition, and how other shapes could be a parallelogram. Each time when he explained his reasoning for a shape, he referred to the definition: "my definition of parallelogram is a shape with three or more sides, that's why it's a parallelogram." During the discussion, he learned that what he had thought was a parallelogram actually went under *polygon*. The next step was to find out what a *parallelogram* was. I tried to have him to make his own definition of parallelogram. I asked him what he thought what a parallelogram was. His answer was at first very strong but not decisive: "My definition of parallelogram would be a shape with at least two parallel sides." I needed to understand how deeply he was aware of his definition. Ogrenci responded to the question if a trapezoid (I showed him a trapezoid) was a parallelogram saying, "... They (*other two sides of trapezoid*) don't have to be because my definition of parallelogram is, at least two or more parallel sides, these two are parallel, the other two actually don't matter." His higher-level reasoning skill was present and sensible, but his prior knowledge was incorrect. This incorrect knowledge most probably constituted by him got in the way of his ability to provide correct statements.

Even though I assessed him at the visual level according to the performance he showed during the tangram problem and sorting-shapes-task, from this particular example I concluded that he was capable of understanding that collection of properties defined classes of figures. (His answer was mathematically partially acceptable; when he was making an assumption, he referred to his own definition. What he said was true in his own reality.) This showed, according to Clements and Battista (1992), and Van de Walle (1998), that he was at the analytic/descriptive level. Furthermore, from his statements, such as "if a shape has 3 or more sides, it is a parallelogram" I could have concluded that he even showed some traits of abstract level but I inferred this notion from his using of "if-then" statements a few times, which is normal in daily speaking, rather than in a mathematical "reasoning" speaking.

Moreover, according to the descriptors of van Hiele's geometric levels described by Fuys et al. (1988), Ogrenci showed a level 2 thinker's characteristics. They identified that students at the analysis level can describe a class of figures in terms of properties and tell what shape a figure is, given some certain features. On the other hand, they cannot explain how certain properties are interrelated. Ogrenci gave the all properties of rectangles, squares, and parallelograms, some of which were redundant, but he could not explain how rectangles are parallelograms, or how squares are related to rectangles (see Figure 4.3). Then the question "Can you draw a square which is NOT a rectangle or a rectangle which is NOT a parallelogram?" led him to think deeply about the interrelationships between the subcategories.

*Attempts to address the problem with instruction.* He was reasoning at a higher-level but his judgments were still intuitive and based upon his incorrect prior knowledge rather than upon the network of relations. This prior knowledge prevented him from giving correct statements. To show the relationships among parallelogram, rectangle, and rhombus, I wrote "parallelogram" and asked to him how he could place rectangle and rhombus under parallelogram. He separated parallelogram into two parts, and labeled them as "rectangle" and "rhombus," drawing an example for each (see Appendix). I needed to understand whether he had grasped the interrelationships between rectangle and rhombus; therefore, I asked him, drawing two Venn Diagrams overlapping each other, to place one shape in each region (see Appendix). After he reasoned, saying, "A square is a rhombus because it has four equal sides, and is a rectangle because opposite sides are parallel and it has four right angles," I asked to him if he could draw a square that was associated with his explanation starting from a given point of it (see Figure 4.2).

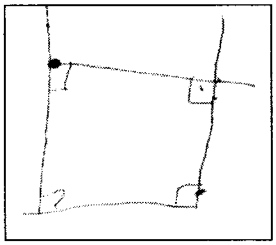


Figure 4.2: Making square.

*A contrasting example.* I will give a contrasting example, which will demonstrate his lower level reasoning for some problems. While trying to resolve his parallelogram-rhombus mismatch, Ogrenci came up with the properties of the figures, giving some features within shapes. He listed all properties of the figures he could think of, i.e., all sides are equal; they have right angles; or opposite sides are parallel, whereas, when asked the specific properties of the figures, he could not see what specific features they had. He reasoned for his definition of rhombus as "two parallel and two diagonal sides facing the same way" without understanding the meaning of "facing the same way."

When we look at Ogrenci's reasoning about rhombus (see Appendix), he could not see the relationships between the properties that he had identified. This showed that he was at the analysis level of geometric thinking. He specified the properties of the figures (see Figure

4.3). However, Cathcart et al. (2001) state that at this level "they [students] don't see any relationships between any of the properties. Since they don't realize that some properties imply others, children's thinking at this level will list every property of a class that they can think of" (p. 295).

Generally, students just memorize the rules or definitions without understanding any underlying meaning(s) of them, and they use/repeat these "memories" like a "parrot." This might be a "parrot learning" (... they are diagonal lines facing the same way). Furthermore, they may not understand the meaning of "facing the same way." This claim cannot be generalized but the individual examples show that students learn (*memorize*) the facts without grasping their meanings. It seemed that Ogrenci was doing exactly this. Van de Walle (1998) emphasizes rote learning and superficial success, and, referring to Crowley (1987) and to Fuys, Geddes, and Tischler (1988), he subscribes that "a student may memorize *the facts or* geometric proofs but fail to create the steps or understand the rationale involved" (p. 347, italics are added).

*Attempts to solve the problem with instruction.* At this point, to solve Ogrenci's terminology problem, I prepared more example/non-example activities regarding squares, rectangles, and parallelograms (see Appendix). With these activities, I aimed at his understanding of specific shapes such as square, rhombus, etc., and their properties individually. To dig out the relationships between squares, rectangles, and rhombus better, I used the analogy problem, which was animal classification. (Flying animals and mammals; birds and flying animals [a bird not flying]; overlapping regions and excluding regions).

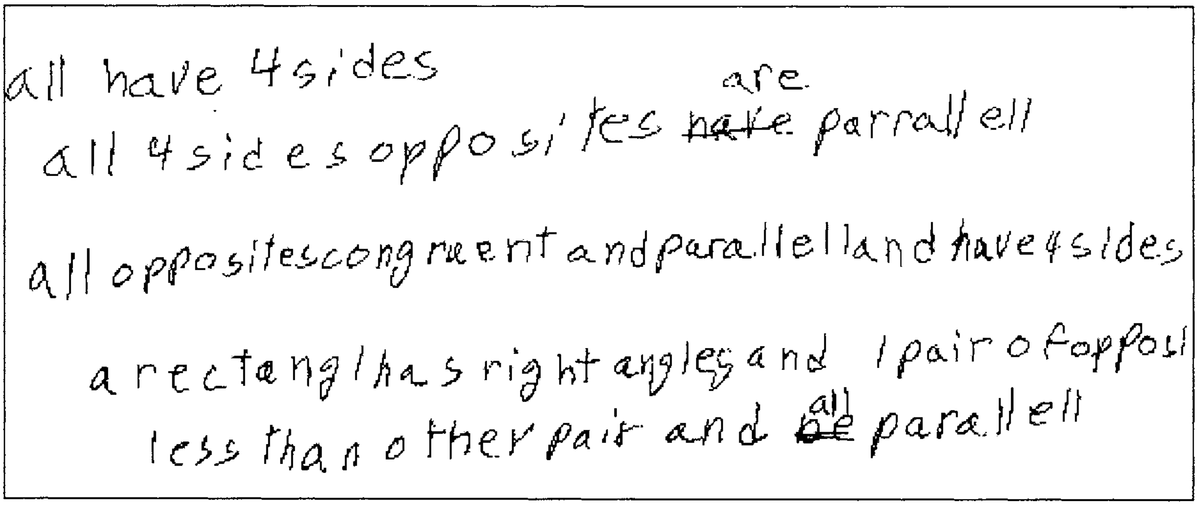



Figure 4.3: Parallelogram-example/non-example task results. He listed all the properties of the shapes, but not the relationships between them.

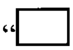
## General Results

From this particular case study, I found 3 fundamental concerns, some of which have branches. First, Ogrenci had a strong negative attitude toward mathematics that seemed to me hard to change, which was constant through the levels. Second, Ogrenci's reasoning skills were changeable through the van Hiele levels. Under this result, Ogrenci showed reasoning skills at the higher levels but gave inaccurate statements due to his incorrect prior knowledge. Third, he showed different geometric thinking of van Hiele levels across the topics we worked on.

# CONCLUSIONS

In this study, I sought to come to know a sixth grade student's understanding of geometric concepts, based on van Hiele levels of geometric thought; whether his/her attitude toward mathematics changes through the levels; and how the learning sessions we had reflect constructivist view of learning to van Hiele levels and geometry. This last section focuses on three basic concerns drawn from the results of this case study: (1) the usefulness of the van Hiele levels of geometric thought and phases of instruction; (2) the impact of relationships between affective issues and mathematics achievement; and (3) how constructivism impacts the van Hiele phases of instruction and geometry.

## Implications for the van Hiele Levels

My research shows that it is not clear how one can determine the level of an individual who knows the rules or definitions by heart for a particular geometric shape when definitions are incorrect or when they are not understood. For example, " and  are most alike because they have four sides that is what Mr. X told us" (level 1 or level 2 thinker?). The same individual may respond the question "Which two are most alike, , , ?" as "first two because opposite sides are parallel and congruent." In the second example, he/she gives the properties of the shapes and even the relationships between shapes (level 2 thinker?). Thus, can we say, "Levels are related to the experience that an individual has had with a particular geometric figure"?

Further, my research concludes that students can reason at the higher levels but still get incorrect statements if their prior knowledge is not correct. This incorrect prior knowledge prevents students' ability to give correct statements. Ogrenci showed time and again, that his level of reasoning was identifiable, but that his incorrect prior knowledge, or the fact that he probably had not constructed that prior knowledge for himself, got in the way of new learning. It is not clear that how reasoning skills affect one's geometric thinking levels, which is also related to his/her using correct mathematical contents and words. My research shows that students can reason at the higher level using inappropriate mathematical content and language. However, to address this issue, there is need for more research studies conducted focusing on students' reasoning skills across different van Hiele levels.

In addition to the first two conclusions, the range of Ogrenci's transition between two consecutive levels was unpredictable. He showed the characteristics of the other levels at one level ranging the former and latter levels. Additional research needs to be done to explore how wide is the transition between consecutive levels.

## Implications for Affective Issues

Another crucial point that I concluded from my research is that my participant's emotions, attitudes, and beliefs about mathematics get in the way of his learning mathematics. The *misperceptions* about mathematics and mathematicians affect children's learning of mathematics. Ogrenci verbally did not express that he was thinking mathematician as a "machine" but I inferred that he was thinking mathematicians as the one who could make practical computations, and mathematical problems easily. An example of such a perception is a mathematician seems to be a "machine" that is able to perform the four computations perfectly. This misperception often drives from the traditional emphasis on procedural knowledge in school mathematics (e.g., basic facts). These issues generally form the basis of belief systems; consequently, students should have a more realistic picture of mathematics during the school years. Teachers should be aware of children's emotions, attitudes, and beliefs that affect their learning of mathematics. If teachers are aware of students' affective status, they can take precautions to solve the problems in advance, especially in the early grades. On the other hand, to better understand the interaction between affective issues and mathematics achievement, further study is needed with more participants, even across different mathematical content.

**Implications for Constructivism**

Even though this is the result of interaction with one student, it does suggest that the phases of instruction may need a moderate revision. Ogrenci was the center of the learning periods during the sessions, in which he had the opportunity to use his previous knowledge to make comments on the problems. Although he had incorrect knowledge about the context, he throughout the interaction with me corrected that wrong knowledge. My research offers additional support to the existing research about the van Hiele phases of instruction that focuses too much on the job of the teacher. Throughout the phases of instruction, the teacher should have students figure out how they know their spontaneous knowledge and how they can transfer this knowledge to the new situation. In this way, the teacher will not only include students in every phase of instruction but also have the opportunity to learn his/her students' misconceptions or incorrect knowledge. Another suggestion to address this issue would be that teachers should have enough background about students' geometric learning and about constructivism.

Students must learn how to develop their discussion on a particular situation rather than learning how to answer the teacher's questions. Instead of giving the definitions or rules to the students, we should have them to find the definitions and rules using their reasoning skills. With well-prepared instruction, the teacher can pose questions that unearth students' thoughts.

Another role of the teacher appears during concept learning in mathematics. For the conceptual learning, teachers should make sure that their students abstract the concept and use in appropriate situations. Teachers can also facilitate students represent the content in different modes and have them translate one form to another. This may be a graph, a chart, a written explanation, a verbal presentation, or a real world situation. In this way students can comprehend their sense of the concept.

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# APPENDIX

## FURTHER SAMPLES OF DATA ANALYSIS TECHNIQUES

### Episodes for Fraction Representations

We had two sessions on fractions at the beginning. The first session focused on the participant's understanding of fractions, and part-whole relationship. In the second session, the participant was asked to fit the tangram pieces into the empty square that the parts had been cut from.

*Fraction Episode 1*

I prepared two fraction problems for the first study: to understand his knowledge about what fraction meant for him, and to get an idea of his understanding of the part-whole relationship.

During the first activity I posed questions about "estimating the fraction of a taken part of a whole," "showing fractions on a line, with manipulatives," and "comparing two fractions." I asked to him how he could estimate the shaded part of the shape "" with fractions, and where this fraction could fit on the fraction line. He said, "I can find it overlapping the shaded part around the bigger shape, and this would be less than 1/2 but I am not quite sure whether it is less than 1/4 or not." I had him to cut the shaded piece and do what he had said. He found the shaded part was less than 1/4 because when he overlapped the piece four times, he saw some extra region left. This was his reasoning for this problem.

Ogrenci was also asked to place the fractions 2/3 and 3/4 on the number line. When I asked him to explain why he had put them where he had placed, he said, "2/3 is 66% and 3/4 is 75%, and since 75% greater than 66%, and 66% is greater than 50%, which is 1/2, I put them in this order." I then asked him to show 2/3 and 3/4 on the rectangle. His answer was he did not know how to divide rectangles, he only knew how to divide circles. Considering his answer, I had him to show with circles, and asked to him how he thought when he was doing with circles. After he explained how he had done, I said, "Why don't you think in the way in which you thought when you were doing with circle?" This pushed him to analyze his own thinking.

The second activity was for understanding part-whole relationship. Two types of questions were asked: "If  is 3/4 of a cake, what is the whole?" and "If  is 4/3 of a cake, what is the whole?" With his reasoning for these questions, Ogrenci seemed to me he had grasped the idea behind part-whole relationship (e.g., the numerator is "how many pieces we have", and denominator is "how many pieces one whole is divided into"). He divided the shapes according to the numerators, which he stated as what he had, and then he marked the number of pieces according to what the denominators were.

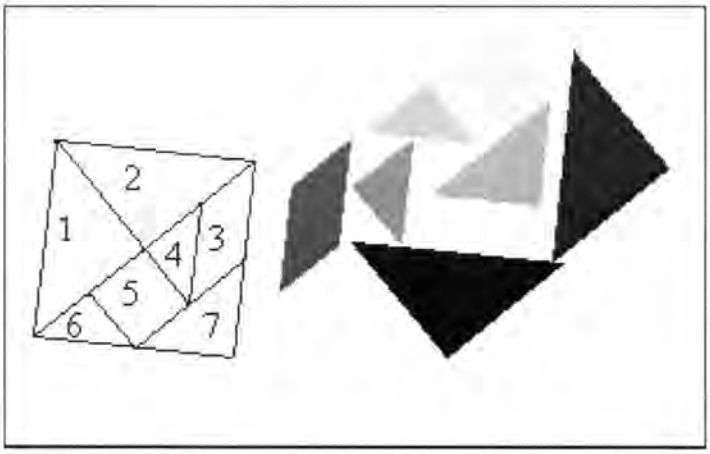


Figure A.1: Tangram

*Fraction Episode 2: Tangram Problem*

Ogrenci was to redesign the given tangram pieces. He had many attempts to do this activity using his vision. He sometimes had some parts leftover, and next time he started with that piece. After he got the original shape, he was ready to find out the relationships between tangram pieces. I wanted him to choose a piece as a unit (he chose a little square, #5) and find the other parts using the piece he chose as a unit. When finding the other pieces, he used the triangles (#4 and 6) matching together, and then converted to the square (i.e., #5 is equal to #3 because using #4 and #6, I can make both #3 and #5, I mean, #4 + #6 = #5, and #4 + #6 = #3 so then #5 = #3).

Having him to find all parts, I asked him to change his unit. He had one of the little triangles (#4 and #6) because he said that he had used those little triangles to find others, so it was easier. After doing all these activities, I suggested him to choose the big square as a unit and to write the other pieces according to the big square. Starting from the bigger pieces, he identified all parts using the big square as a unit.

## Episode for Learning Period Between Level 2 and Level 3

*Level 2: Analytic/Descriptive*

Analyzing shapes on the basis of their properties, focusing on the parts of the shapes, and understanding that collection of properties goes with classes of figures are expected thoughts from children.

*Learning Period*

Students move from level 2 to level 3 understanding the characteristics of shapes and relationships between properties of figures and among classes of figures. The main objectives of this learning period between level 2 and level 3 are networks of relationships and the ordering of properties / rules of geometric shapes.

*Phase 1: Information/Inquiry.* Students have the opportunity to construct figures based on properties, and relationships.

Ogrenci was asked to draw the same tangram shape by looking at the outline. He first tried to do the big square, and then went through the pieces. He explained how he had reached that shape.

Recai: Okay, can you explain how you've got this shape?

Ogrenci: I first drew the big square because it was given as whole. I knew that we have two triangles that are one-forth of the square [when we were working on the tangram problem he had found the ratios].

Recai: How did you get the big triangles?

Ogrenci: I divided the square into four triangles like we did before [we were laughing because we had had that discussion], and these two are our triangles.

Recai: What about this parallelogram, rhombus with your saying? We'll discuss that later.

Ogrenci: Well, actually, that one was left over, I mean I drew the other parts, and I got this automatically.

His using the word rhombus for parallelogram pushed me to understand what he meant by rhombus. From the discussion we had, he had said that rhombus was a specific type of *parallelogram* (his definition of parallelogram was still a shape with three or more sides). That was true because indeed rhombus is a polygon. On the other hand, he could not specify those properties that a rhombus has. These conflicts were my next concerns through the next phases.

*Phase 2: Guided Orientation.* Students discover the properties of figures with the materials related to the content.

I needed to take him from where we left to promote his understanding of polygons. Since we already did sorting-shapes-task, I asked him which of those shapes would be parallelogram according to his definition. Below is the episode that we had a discussion on polygons and parallelogram.

*Episode for Polygons and Parallelogram*

Instead of giving the definitions or rules to students, we should have them to find the definitions and rules using their reasoning skills. With well-prepared instruction, the teacher can pose questions that unearth the students' thoughts.

Recai: Okay, you are still saying that parallelogram has 3 or more sides?

Ogrenci: Yes, my definition of parallelogram, mm… if a shape has 3 or more sides, it's a parallelogram.

Recai: Let's change the name of the definition and give another name. What would you name the definition instead of parallelogram?

Ogrenci: This? So then "an object" that has 3 or more sides.

Recai: Ok. My logic says if they gave this name to this definition, it should have logical things that specify the shape. Why did they name this a parallelogram? What do you think?

Ogrenci: mmm… I don't know but my definition of parallelogram is this, 3 or more sides.

Recai: Okay, my understanding says if they named parallelogram, the shape must contain some parallel features. And when I look at this shape, rhombus as you defined, these two lines (or sides) are parallel, so are these two sides. You're saying this hexagon is parallelogram, right?

Ogrenci: Yeah, because it has more than three sides.

Recai: But where is the parallelism?

Ogrenci: These two sides, these two sides and these two sides should be parallel.

Recai: So... what if they are not parallel? What about this shape (I drew a pentagon none of whose sides can be parallel). [Pentagon]. Why didn't you say parallelogram?

Ogrenci: yeah, still parallelogram, it has specific name.

Recai: But, you were saying that these sides should be parallel? Can you see, here, parallelism?

Ogrenci: No, but my definition of parallelogram, if it has 3 or more sides, it's just, mm...

Recai: Ok, have you heard polygon?

Ogrenci: T.V. show or something, I don't know what the polygon is.

Recai: What if I said a polygon is a shape that has 3 or more sides?

Ogrenci: That'd be what I think a parallelogram is then, so... so, this definition goes under polygon not parallelogram.

Recai: So, we have three things...

Ogrenci: Yeah, polygon, parallelogram, and rhombus.

Recai: Now, you've changed the definition of rhombus? What do you thi…

Ogrenci: It's actually a specific shape of polygon, so then what is the definition of parallelogram?

Recai: What do you think?

Ogrenci: A parallelogram is something with parallel lines, … a shape of parallel lines.

Recai: How many parallel lines do you think?

Ogrenci: mm… probably two, 'cause you can't think all of them parallel, it wouldn't be a shape, just a bunch of lines.

Recai: Okay, now you've learned that your definition of parallelogram was actually polygon. How would you define parallelogram then?

Ogrenci: My definition of parallelogram would be a shape with at least two parallel sides.

Recai: Is this a parallelogram (I showed the trapezoid that I had drawn).

Ogrenci: Yeah, these two lines are parallel.

Recai: What about other two? Do they also have to be parallel?

Ogrenci: They don't have to be because my definition of parallelogram is at least two or more parallel sides, these two parallel, the other two actually doesn't matter.

To get him to understand the concept of parallelogram (i.e., a four-sided shape with the opposite sides parallel – actually no need for using the term quadrilateral even though it is more general than parallelogram, and not really necessary to see opposite angles are also congruent), I gave him parallelogram-example/non-example task (see Figure A.2). The results of this task can be seen in Figure 3.6 in the Result section.

After Ogrenci's understanding of parallelogram, the next procedure was to progress in understanding the more specific term rhombus through Phase 3: Explication, where students become aware of the relationships, try to explain them in their own words, and learn technical language for that subject matter with the material provided by the teacher.

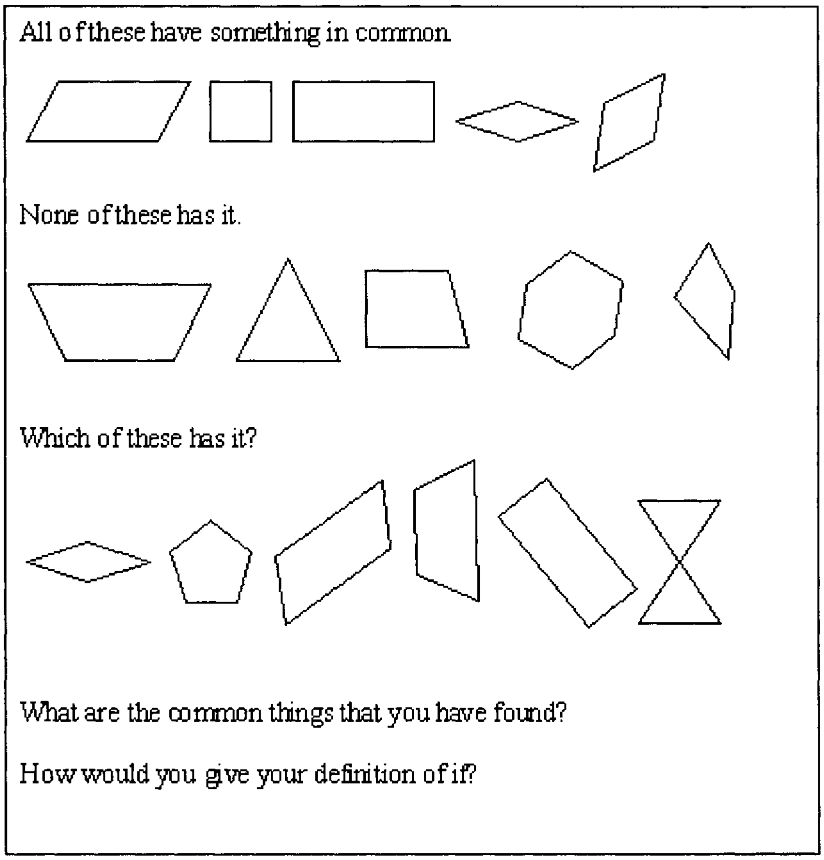


Figure A.2: Parallelogram-example/non-example task

In the discussion about rhombus, we proceeded to the some properties of rhombus using his understanding. Before the parallelogram-example/non-example task, rhombus had been a shape of *parallelogram* (polygon) that had two parallel and two diagonal sides, and diagonal sides did not have to be parallel. We did another example/non-example activity (see Figure A.3) to check for the concept development.

The results of example/non-example tasks (see Figure 3.6 in Result section) directed me to find out how he would see the relationships among parallelogram, rectangle, square, and rhombus, which seemed more problematic. Through Phase 4: Free Orientation, students try to gain experience within explicit network of relations among the objects of the structures being studied with more open-ended activities that can be solved by several different solutions.

I asked him what he thought where squares could go among those shapes. At first he did not see the relationships, but after I asked what he thought the relationship between flying animals, penguins, and birds, "penguins are birds but they cannot fly" he said. As can be seen in Figure 3.6 (Result section), he gave all the properties of shapes, but he could not explain how these shapes were interrelated as he did with animals.

The questions "Can you draw a square which is NOT rectangle or a rectangle which is NOT parallelogram?" had him to think deeply the interrelationships between the subcategories, which would be dug out through the Phase 5: Integration, where students summarize all they have internalized about the content and then integrate their knowledge into the newly formed network of relations.

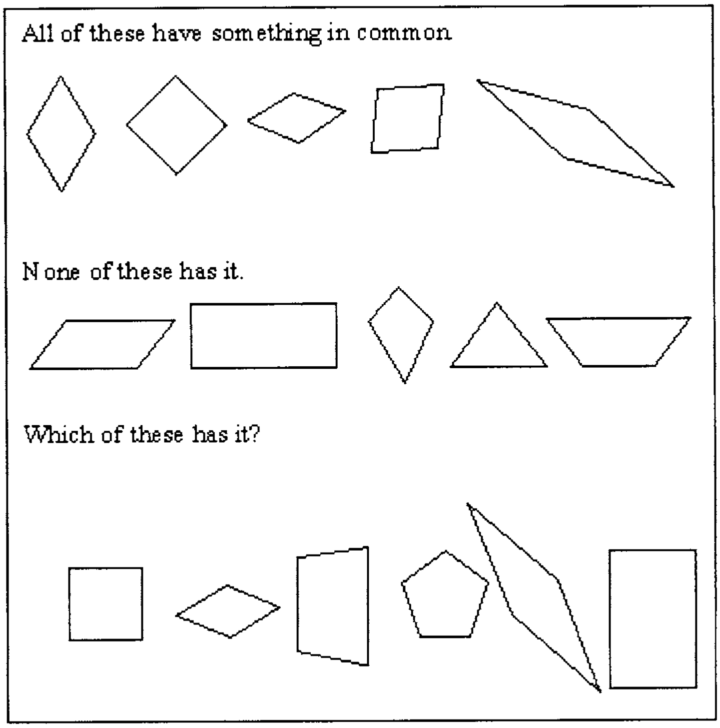


Figure A.3: Rhombus-example/non-example task (adapted from Van de Walle (1998)). "The name of a property is not necessary for it to be understood. It requires more careful observation of properties to discover that shapes have in common" (p. 363).

The first interrelationship was between parallelogram, rectangle, and rhombus. He separated parallelogram into two parts and labeled as "rectangle" and "rhombus" drawing an example for each (see Figure A.4). Then I asked him to explain his drawing. He answered, "since the opposite sides of rectangle and rhombus are parallel, they are also parallelogram." I needed to understand whether he grasped the interrelationships between rectangle and rhombus; therefore, I drew two Venn Diagrams overlapping each other, and asked to him, "If you place one shape in each region, what would you put in them?" He put a rectangle and a rhombus in each excluding regions consecutively, and wrote sq (square) in overlapping region (see Figure A.5). His explanation was simple but strong: "Square is a rhombus because it has four equal sides, and is a rectangle because opposite sides are parallel and it has four right angles."

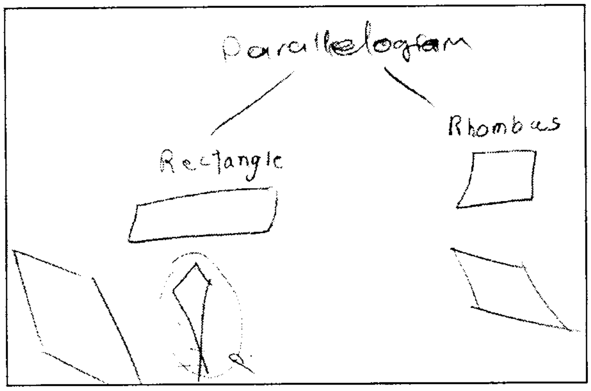


Figure A.4: Classifying parallelogram.

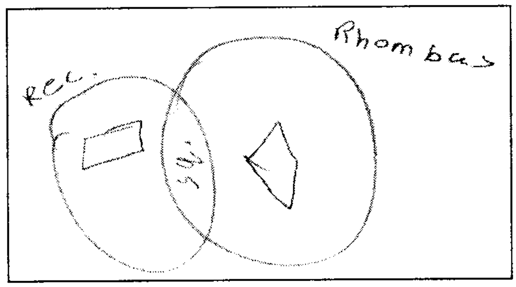


Figure A.5: Intersection of rectangle and rhombus.

For specific shapes such as parallelogram, rectangle, square, and rhombus, he could find the relationships regarding their properties within and among groups. Thus, he was ready to proceed to the next level, abstraction/informal deduction, where students are more aware of interrelationships within and among classes of shapes, and start to give informal deduction.

**Episode for Learning Period Between Level 3 and Level 4**

*Level 3: Abstract/Informal Deductive*

Having internalized relationships between classes of figures and between properties of shapes, students are able to engage in "if-then" reasoning and classifying shapes with minimum characteristics. They more focus on the reorganization of ideas and interrelationships among properties of geometric shapes (Clements & Battista, 1992; Van de Walle, 1998).

*Learning Period.* Students move to formal deduction level through this learning period experiencing interrelationships between and among geometric shapes using informal deduction.

In this study, my participant was able to access level 3, abstraction/informal deduction, through the sessions with limited characteristics of this level. However, I will give a brief episode to show what kind of characteristics he performed regarding abstraction level.

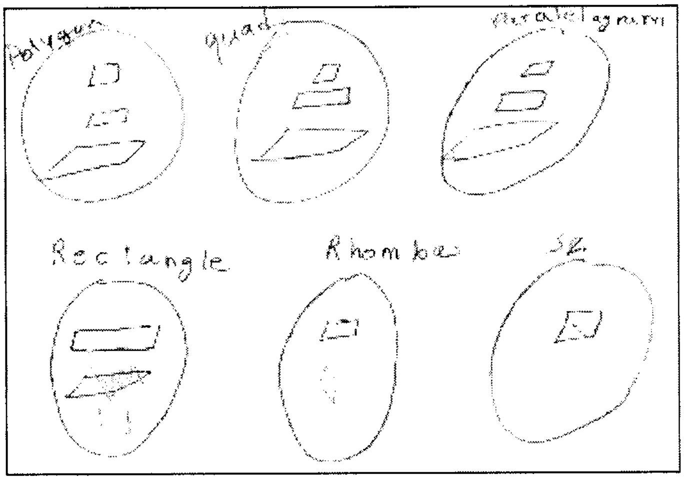


Figure A.6a: Polygons in diagrams (First row, from top to bottom: Square, rectangle, and rhombus; second row, from top to bottom in *rectangle*: rectangle, rhombus (scratched), and square.

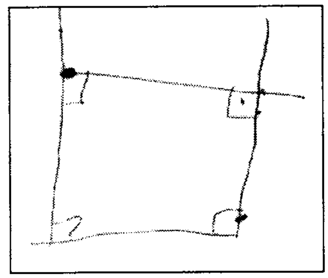

Figure A.6b: Making square.

Having seen he could make the connection between parallelogram, rectangle, square, and rhombus, I asked him to do the same thing with all other geometric figures. He drew six Venn Diagrams, labeled them from polygon to square (with my help), and put some shapes into them (see Figure A.6a). As seen in the figure, he drew square, rectangle, and parallelogram in the first three diagrams (*polygon, quadrilateral,* and *parallelogram*); rectangle and square under *rectangle*; square and rhombus under *rhombus*; and square under *square*. Italics refer to the name of diagram. From this drawing, I asked to him what he concluded and what it meant for him. He explained;

> since square, rectangle, and parallelogram have same characteristics as these three (*polygon, quadrilateral, and parallelogram*), I put them over there; rectangle and

square over here (*rectangle*); square and rhombus in *rhombus* diagram; and square for *square*.

He further said, "If square is in both *rectangle* and *rhombus*, then it should be in this region" (intersection of *rectangle* and *rhombus* diagrams). I asked to him if he could draw a square starting from a given point of it (see Figure A.6b). He gave reasons for steps in drawing the square. As seen in the figure, I put the first point, and he completed the other parts of the square. He first put another point to the right of the first one, and drew a line, then he put the third point from which he drew another line to the second point, and made right angle, and then he finished the others in the same way saying, "all four sides are equal and angles are 90 degrees."

Recai: So, what does it tell you?

Ogrenci: All sides are equal, so it is a rhombus, and opposite sides are parallel and it has right angles, so it is a rectangle.

Recai: How can you be sure they are parallel?

Ogrenci: They have right angles, they never touch.

Since this was the last session we met according to the agreement, we had no further activity regarding mathematics. We then began to talk about how Ogrenci thought about this study, and how the activities were different than those he currently had.

When we were going home, we had the conversation below.

Recai: Do you know you have used mathematics when you were going out from this building?

Ogrenci: No way, how?

Recai: Think of what you have done!

Ogrenci: I turned left and right, and I found the way.

Recai: That's why I said you are a mathematician.

Ogrenci: No, I am not, and I don't want to be!

Recai: Yes, you're. Even though you don't like mathematics, you have some mathematics sides in you. [There is not].

Recai: At least you think mathematically, even now. [He was looking at outside from the car, and saying something in his mouth. Maybe he was thinking I am one of those crazy mathematicians].

Recai: What do you say? You didn't use math? Think again what you have done.

Ogrenci: Turn left and right, and find the way.

Recai: Can you see any connection to math?

Ogrenci: Oh, yeah, inverse relationship or equation or something like that, crab!!!!